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GROUPES DE LIE ET PUISSANCES RÉDUITES DE STEENROD.*

Par A. Borel et J.-P. Serre.

Introduction. N. E. Steenrod a défini de nouvelles opérations cohomologiques, les puissances réduites, qui généralisent ses i-carrés. Nous nous proposons ici d'étudier ces opérations dans la cohomologie mod p, (p premier), des groupes de Lie et de leurs espaces classifiants, et d'appliquer les résultats obtenus à divers problèmes.

Pour la commodité du lecteur, nous avons rappelé dans la première partie tous les principaux résultats sur les groupes de Lie et leurs espaces classifiants dont nous avons à faire usage par la suite, en les complétant du reste sur quelques points. La deuxième partie est consacrée aux puissances réduites, dont nous indiquons les propriétés au No. 7, sans en répéter la définition explicite, qui n'interviendra pas ici; nous calculons ensuite ces opérations dans les espaces projectifs complexes et quaternioniens, ce qui permet d'obtenir quelques renseignements sur les groupes d'homotopie des sphères, par la méthode de Steenrod.

Dans la troisième partie, nous combinons les résultats de I et II pour étudier les puissances réduites dans les algèbres de cohomologie $H^*(G, Z_p)$ et $H^*(B_G, Z_p)$ d'un groupe de Lie compact connexe G et d'un espace B_G classifiant pour G, *lorsque G et son quotient G/T par un tore maximal sont sans p-torsion.*[1] Dans ce cas en effet, $H^*(B_G, Z_p)$ s'identifie à une sous-algèbre de $H^*(B_T, Z_p)$; or $H^*(B_T, Z_p)$ est engendrée par ses éléments de degré deux, (B_T peut même, si l'on veut, être envisagé comme produit d'espaces projectifs complexes), et les puissances réduites y sont donc connues d'après II. Cela détermine en principe les puissances réduites dans $H^*(B_G, Z_p)$, et par conséquent aussi dans $H^*(G, Z_p)$ car, sous les hypothèses faites, $H^*(B_G, Z_p)$ est une algèbre de polynômes dont les générateurs sont images par transgression de générateurs de $H^*(G, Z_p)$, (qui est une algèbre extérieure), et la transgression commute aux puissances réduites. Cette méthode générale est ensuite appliquée aux groupes unitaire $\boldsymbol{U}(n)$ et unitaire symplectique $\boldsymbol{Sp}(n)$ pour p (premier) quelconque, au groupe orthogonal

* Received February 22, 1953.

[1] On dit qu'un espace a de la p-torsion (p premier), si l'un de ses groupes d'homologie entière a un coefficient de torsion divisible par p.

$\boldsymbol{SO}(n)$ pour $p \neq 2$, et à leurs espaces classifiants, ce qui donne, entre autres, des résultats sur les classes de Chern.

La quatrième partie est consacrée aux applications. Nous montrons dans le No. 15 la non-existence de structures presque-complexes sur $\boldsymbol{S}_n$, $(n \geq 8)$, et d'un type d'algèbres à division de dimension > 8, et dans le No. 17, la non-existence de sections dans certaines fibrations, (par exemple $\boldsymbol{U}(n)/\boldsymbol{U}(n-1) = \boldsymbol{S}_{2n-1}$) ; on en déduit quelques renseignements sur les groupes d'homotopie des groupes classiques, auxquels nous consacrons également les Nos. 18 et 19. Enfin, le No. 20 donne des conditions nécessaires pour l'existence de sections dans des fibrations où espace, base et fibre sont des variétés de Stiefel complexes.

I. Espaces fibrés à groupe structural de Lie.

1. Espaces universels et espaces classifiants pour un groupe de Lie. Soit G un groupe de Lie compact; rappelons que l'on appelle *espace universel pour* G jusqu'à la dimension n un espace E, fibré principal de groupe structural G, tel que $\pi_i(G) = 0$ pour $0 \leq i \leq n$, ([14], § 19) ; sa base $B = E/G$ est dite *espace classifiant pour* G et pour la dimension n, elle permet en effet de classer tous les espaces fibrés principaux de groupe structural G ayant comme base un polyèdre X donné de dimension n, (une classe d'espaces fibrés correspondant biunivoquement à une classe d'applications homotopes de X dans B, *voir* [14], § 19).

On sait, ([14], 19. 7), que l'on peut trouver pour tout G et pour tout n des espaces universels qui sont, ainsi que leurs bases, des variétés analytiques compactes, donc des polyèdres finis; il est souvent commode de les considérer pour n arbitrairement grand et pour éviter d'avoir à préciser cet entier, on peut introduire les notions d'espace universel et d'espace classifiant pour G, (sous-entendu: pour tout n), de la façon suivante:

Soit $E_1, E_2, \cdots$ une suite d'espaces universels pour G et pour des dimensions $n_1 < n_2 < \cdots$; il est clair que l'on peut la choisir telle que E_i et $B_i = E_i/G$ soient des polyèdres finis et qu'il existe un homéomorphisme f_i de E_i dans E_{i+1} commutant avec les opérations de G, $(i = 1, 2, \cdots)$. La limite inductive E des E_i est alors un espace fibré principal de groupe structural G, dont tous les groupes d'homotopie sont nuls; E sera dit universel pour G, sa base B, qui est limite inductive des espaces B_i, sera dite espace classifiant pour G. On déduit immédiatement du théorème de classification que deux espaces universels, ou deux espaces classifiants, ont même type d'homotopie; il n'y a donc pas d'inconvénient, tant que l'on ne s'intéresse qu'à

des questions homologiques ou homotopiques, à désigner par E_G, resp. B_G, l'un quelconque de ces espaces; en particulier, on pourra parler des groupes d'homologie singulière ou de cohomologie singulière de l'espace B_G; on a d'ailleurs, Γ étant un groupe de coefficients:

$$H_p(B_G, \Gamma) \approx H_p(B_i, \Gamma), \qquad H^p(B_G, \Gamma) \approx H^p(B_i, \Gamma) \qquad \text{si } n_i > p,$$

(les isomorphismes en cohomologie étant bien entendu compatibles avec le cup-produit si Γ est un anneau); les groupes $H^p(B_G, \Gamma)$ sont donc isomorphes aux groupes $H^p(B_i, \Gamma)$ lorsque i est assez grand, on retrouve ainsi les conventions de [2], § 18.[2]

Nous avons donc fait correspondre à tout groupe de Lie compact G un espace, ou plutôt une classe d'espaces ayant le même type d'homotopie, B_G. On peut de plus associer à tout homomorphisme $f: H \to G$ d'un groupe de Lie compact dans un autre une classe d'applications homotopes $\rho(f): B_H \to B_G$.

Soit en effet E_H un espace universel pour H; par extension du groupe structural de H à G au moyen de f, E_H définit un espace E'_H principal pour G et de base B_H; cet espace est le quotient $(E_H, G)_H$ du produit $E_H \times G$ par la relation d'équivalence $(x, g) \approx (x \cdot h, f(h) \cdot g)$. Comme la base de E'_H est limite inductive de polyèdres, le théorème de classification s'applique, et il existe une classe d'applications homotopes $\rho(f)$ de B_H dans B_G telle que E'_H soit l'image réciproque de E_G par les $\rho(f)$; cela définit les $\rho(f)$.

Bien entendu, $\rho(f)$ est homotope à l'identité si f est l'identité, et les $\rho(f)$ vérifient la propriété de transitivité $\rho(f \circ g) = \rho(f) \circ \rho(g)$. On peut donc dire que la correspondance $G \to B_G$ est un *foncteur covariant* de G.

Le cas particulier le plus important de la notion précédente est celui où f est l'application identique d'un sous-groupe fermé H d'un groupe G dans le groupe G lui-même; on désigne alors $\rho(f)$ par $\rho(H, G)$, conformément aux notations de [2], § 21. Dans ce cas on peut choisir $\rho(f)$ de telle sorte que $\rho(f): B_H \to B_G$ *définisse B_H comme espace fibré de base B_G et de fibre l'espace homogène G/H.* En effet, puisque H est plongé dans G, il opère sur E_G et E_G, muni de ces opérateurs, est un espace universel pour H, que l'on peut prendre comme E_H. On aura alors $B_H = E_G/H$, $B_G = E_G/G$ et l'application $\rho(f)$ n'est autre que la projection canonique de E_G/H sur E_G/G; elle définit bien B_H comme espace fibré de fibre G/H et de base B_G.

Dans le cas général, on peut, au moins au point de vue homologique ou

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[2] Si G est discret (cas que nous n'avons pas exclu), les groupes $H_p(B_G, \Gamma)$ et $H^p(B_G, \Gamma)$ ne sont autres que les groupes d'homologie et de cohomologie de G au sens de Hopf-Eilenberg-MacLane-Eckmann.

homotopique, envisager aussi $\rho(f)$ comme projection d'un espace fibré. Soit en effet $X = (E'_H, E_H)_H$ le quotient de $E'_H \times E_H$ par la relation d'équivalence $(x, y) \approx (x \cdot h, y \cdot h)$, il admet deux fibrations, l'une de fibre E'_H et de base B_H, l'autre de fibre E_H et de base B_G; notons α et β les projections correspondantes. Comme E_H est acyclique, β définit un isomorphisme β_* des groupes d'homologie (ou d'homotopie) de X sur ceux de B_H. D'autre part un homomorphisme de E'_H dans E_H définit de façon évidente une section s: $B_H \to X$, qui composée avec α redonne $\rho(f)$; ainsi, une fois les groupes d'homologie de B_H identifiés à ceux de X par ${\beta_*}^{-1}$, l'homomorphisme $\rho_*(f)$ devient l'homomorphisme α_* induit par la projection α.

En particulier, si f est la projection de H sur son quotient H/N par un sous-groupe invariant fermé N, l'espace E'_H est visiblement un espace B_N et $\rho_*(f)$ *s'identifie à l'homomorphisme induit par la projection d'un espace fibré ayant même homologie que* B_H, *de fibre* B_N *et de base* $B_{H/N}$.

Note. Soit E un espace fibré de groupe structural H dont la base X est un polyèdre fini; E est donc bien défini par une classe d'applications homotopes $\zeta: X \to B_H$; en les composant avec les applications $\rho(f): B_H \to B_G$, on définit donc un espace fibré de base X et de groupe structural G; en outre, d'après la construction même de $\rho(f)$, cet espace est celui que l'on obtient à partir de E *en étendant le groupe structural de* H *à* G *au moyen de* f.

C'est sous cette forme que l'on trouvera étudiée, dans des cas particuliers, l'application $\rho(f)$, notamment par Wu ([18], [19]). On voit également que pour qu'un espace fibré de base X, de groupe G, défini par $\zeta: X \to B_G$ puisse être obtenu à partir d'un espace fibré de groupe H par extension du groupe structural il faut et il suffit que ζ puisse se " factoriser " par $\rho(f)$. Si l'on connait les algèbres de cohomologie $H^*(B_H)$ et $H^*(B_G)$, ainsi que $\rho^*(f)$: $H^*(B_G) \to H^*(B_H)$, on tire de là des conditions cohomologiques nécessaires pour que l'on puisse restreindre le groupe structural de G à H. C'est là la méthode suivie par Wu [18] pour étudier les structures presque complexes, (f étant alors l'inclusion de $\boldsymbol{U}(n)$ dans $\boldsymbol{SO}(2n)$).

2. Cohomologie des groupes de Lie et de leurs espaces classifiants. Soit G un groupe de Lie compact connexe de rang l, (rappelons que le rang est la dimension commune des tores maximaux de G). D'après un théorème classique de Hopf, l'algèbre de cohomologie de G relativement à un corps de caractéristique zéro est une algèbre extérieure engendrée par l éléments de degrés impairs. Ce résultat vaut encore pour $H^*(G, Z_p)$, (p premier, Z_p corps des entires modulo p), lorsque G est sans p-torsion,[1] ([2], Proposition 7. 2);

de même, si G n'a pas de torsion, $H^*(G, Z)$ est l'algèbre extérieure d'un groupe abélien libre ayant l générateurs de degrés impairs.

Dans le cas où G est sans p-torsion, on peut établir des relations très précises entre $H^*(G, Z_p)$ et $H^*(B_G, Z_p)$ en utilisant la transgression. Rappelons que la transgression dans un espace fibré E, de base B et de fibre F, relativement à un groupe de coefficients Γ, est en dimension s, $(s = 0, 1, 2, \cdots)$, un homomorphisme :

$$(2.1) \qquad \tau : T^s(F, \Gamma) \to H^{s+1}(B, \Gamma)/L^{s+1}(B, \Gamma)$$

d'un sous-groupe $T^s(F, \Gamma)$ de $H^s(F, \Gamma)$ dans un quotient de $H^{s+1}(B, \Gamma)$. L'homomorphisme τ est le composé $q^{*-1} \circ \delta$, où δ désigne l'homomorphisme de cobord qui applique $H^s(F, \Gamma)$ dans $H^{s+1}(E, F ; \Gamma)$, et où q^* est le produit de l'isomorphisme de $H^{s+1}(B, \Gamma)$ sur $H^{s+1}(B, b ; \Gamma)$, (b projection de F), par l'homomorphisme $p^* : H^{s+1}(B, b ; \Gamma) \to H^{s+1}(E, F ; \Gamma)$ transposé de la projection (*voir* [2], § 5 ; [11], pp. 434, 457). En particulier nous noterons $T^s(G, \Gamma)$ l'ensemble des éléments de $H^s(G, \Gamma)$ transgressifs dans un espace universel E_G, et qui seront dits être *universellement transgressifs*.

Ces définitions étant posées, on peut exprimer ainsi les propriétés de la fibration de E_G par G, base B_G qui sont établies dans [2], (Théorèmes 13. 1, 19. 1) :

2. 2. *Soit p un nombre premier, et supposons G sans p-torsion. Alors $H^*(G, Z_p)$ possède un système de générateurs $h_1, \cdots, h_l$ de degrés impairs qui forment une base du sous-espace $T(G, Z_p)$ de $H^*(G, Z_p)$ engendré par les éléments universellement transgressifs.*

2. 3. *Le sous-espace $L^{s+1}(B_G, Z_p)$ de $H^{s+1}(B_G, Z_p)$*, (notations de 2. 1), *est égal au sous-espace des éléments décomposables de $H^{s+1}(B_G, Z_p)$*, (c'est à dire au sous-espace engendré par les produits d'éléments de degrés $< s + 1$).

Nous désignerons par $D^j(B_G, \Gamma)$, ou par D^j si cela ne prête pas à confusion, l'ensemble des éléments décomposables de $H^j(B_G, \Gamma)$ et par $D(B_G, \Gamma)$, ou par D, la somme directe des D^j.

2. 4. *Soient τ la transgression dans E_G et $y_i \in H^*(B_G, Z_p)$ un représentant de $\tau(h_i)$, $(i = 1, \cdots, l)$. Alors $H^*(B_G, Z_p)$ est identique à l'algèbre des polynômes admettant les y_i comme générateurs.*

On a ici $\tau(h_i) = y_i \bmod D$ et $H^*(B_G, Z_p)$ est une algèbre de polynômes à l générateurs dont les degrés sont égaux aux degrés des h_i augmentés d'une unité (et sont par conséquent pairs).

Les résultats 2. 2, 2. 3, 2. 4 restent valables si l'on remplace partout Z_p par un corps de caractéristique zéro, sans hypothèse sur G, ou encore si l'on substitue Z à Z_p lorsque G n'a pas de torsion.

3. Relations entre $H^*(B_G, Z_p)$ et le groupe de Weyl de G. Soient T un tore maximal du groupe de Lie compact connexe G de rang l, N le normalisateur de T dans G, et Φ le groupe de Weyl de G, c'est à dire le quotient N/T. On sait que Φ est un groupe fini. D'après les résultats du No. 2, la transgression établit un isomorphisme de $H^1(T, Z)$ sur $H^2(B_T, Z)$, et $H^*(B_T, Z)$ est l'algèbre symétrique libre engendrée par $H^2(B_T, Z)$; toute base $(\xi_1, \cdots, \xi_l)$ de $H^1(T, Z)$ définit donc par transgression un système de générateurs indépendants $(x_1, \cdots, x_l)$ de $H^*(B_T, Z)$ de degrés égaux à deux

Puisque N est une extension de T par Φ, le groupe Φ opère canoniquement sur T, et de ce fait sur $H^1(T, Z)$ donc aussi sur $H^*(B_T, Z)$. On notera I_G la sous-algèbre des éléments de $H^*(B_T, Z)$ invariants par Φ. Il est clair que si $x \in H^*(B_T, Z)$ est tel que $n \cdot x \in I_G$, (n entier), on a aussi $x \in I_G$; par conséquent I_G est *facteur direct* de $H^*(B_T, Z)$ pour la structure de groupe abélien, ce qui permet de considérer $I_G \otimes Z_p$ comme plongé dans

$$H^*(B_T, Z) \otimes Z_p \approx H^*(B_T, Z_p).$$

Cela étant posé, on a ([2], Prop. 29. 2) :

3. 1. *Si G et G/T sont sans torsion, l'homomorphisme*

$$\rho^*(T, G) : H^*(B_G, Z) \to H^*(B_T, Z)$$

est biunivoque et son image est I_G.

3. 2. *Soit p un nombre premier. Si G et G/T sont sans p-torsion, il en est de même de B_G, l'homomorphisme:*

$$\rho^*(T, G) : H^*(B_G, Z_p) \to H^*(B_T, Z_p)$$

est biunivoque, et son image est $I_G \otimes Z_p$, (plongé dans $H^*(B_T, Z_p)$ comme il a été dit plus haut).

Lorsqu'on sera dans les hypothèses de 3. 1, (resp. 3. 2), on identifiera $H^*(B_G, Z)$, (resp. $H^*(B_T, Z_p)$), à I_G, (resp. $I_G \otimes Z_p$) ; cela permettra comme on le verra de ramener beaucoup de questions portant sur B_G aux questions analogues sur B_T, qui est plus simple à étudier. Soient par exemple H et G deux groupes de Lie vérifiant 3. 2, $f : H \to G$ un homomorphisme et cherchons à déterminer $\rho^*(f) : H^*(B_G, Z_p) \to H^*(B_H, Z_p)$. Soient T' un tore maximal

de H, T un tore maximal de G contenant $f(T')$, et $g: T' \to T$ la restriction de f à T'. Elle définit un homomorphisme $\rho^*(g) : H^*(B_T, Z_p) \to H^*(B_{T'}, Z_p)$ dont la restriction à $I_G \otimes Z_p$ est évidemment $\rho^*(f)$; pour connaître $\rho^*(f)$, il suffit donc de connaître $\rho^*(g)$, ce qui est très facile: Soient $\xi_1, \cdots, \xi_r$, (resp. $\xi'_1, \cdots, \xi'_s$), une base de $H^1(T, Z_p)$, (resp. de $H^1(T', Z_p)$. On peut écrire:

$$g^*(\xi_i) = \Sigma\, n_{ij}\xi'_j, \qquad (n_{ij} \in Z_p),$$

et, si x_i, resp. x'_i, est image par transgression de ξ_i, resp. ξ'_i, un polynôme $Q(x_1, \cdots, x_r)$ en les x_i a comme image par $\rho^*(g)$ le polynôme

$$Q(\Sigma\, n_{1j}x'_j, \cdots, \Sigma\, n_{rj}x'_j),$$

(*voir* pour plus de détails [2], § 28, 31 où cette méthode est appliquée dans le cas où f est biunivoque).

Un cas particulier important est celui où f est l'inclusion d'un sous-groupe H de G dans G, *H et G ayant même rang.* On a alors $T = T'$, et $\rho^*(g)$ est l'identité; tenant compte des identifications faites plus haut on voit que *$H^*(B_G, Z_p)$ s'identifie à une sous-algèbre de $H^*(B_H, Z_p)$, elle-même identifiée à une sous-algèbre de $H^*(B_T, Z_p)$.*

Conditions d'application des résultats précédents. Les hypothèses 3. 1 et 3. 2 sont fréquemment vérifiées, en effet:

3. 3. *Si l'algèbre de Lie de G ne contient aucun facteur isomorphe à $\boldsymbol{E}_6$, $\boldsymbol{E}_7$, ou $\boldsymbol{E}_8$, G/T est sans torsion* ([2], Prop. 29. 1).

3. 4. *Pour tout n les groupes classiques $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$ sont sans torsion* ([2], Prop. 9. 1).

3. 5. *Pour tout n et pour tout nombre premier p impair, $\boldsymbol{SO}(n)$ est sans p-torsion,* ([2], Prop. 10. 4).

Ainsi, *à l'exception de $\boldsymbol{SO}(n)$ pour $p = 2$, tout groupe classique est justiciable de* 3. 2. Dans le cas de $\boldsymbol{SO}(n)$, $p = 2$, il convient de remplacer les tores maximaux par des sous-groupes abéliens maximaux de type $(2, 2, \cdots, 2)$, (*voir* [3]).

4. Cas particulier: le groupe unitaire $\boldsymbol{U}(n)$. Soit $G = \boldsymbol{U}(n)$ le groupe des matrices complexes unitaires à n lignes et n colonnes; les matrices diagonales en constituent un tore maximal $\boldsymbol{T}$, $\boldsymbol{U}(n)$ est donc de rang n. Désignant par $\exp(2i\pi\zeta_1), \cdots, \exp(2i\pi\zeta_n)$ les valeurs propres d'une matrice

diagonale, nous prendrons les éléments $\xi_i = d\zeta_i$ comme base de $H^1(\boldsymbol{T}, Z)$ et noterons x_i l'image par transgression de ξ_i dans $H^2(B_{\boldsymbol{T}}, Z)$.

Le normalisateur $\boldsymbol{N}$ de $\boldsymbol{T}$ dans $\boldsymbol{U}(n)$, est l'ensemble des matrices *monomiales* (c'est à dire produits d'une matrice diagonale par une matrice de permutation), et $\Phi = \boldsymbol{N}/\boldsymbol{T}$ est le groupe des permutations des ξ_i ou des x_i; par conséquent I_G est l'algèbre des polynômes symétriques en les x_i; de même $I_{\boldsymbol{U}(n)} \otimes Z_p$ est l'ensemble des polynômes symétriques à coefficients dans Z_p.

Soit C_{2i} la i-ème fonction symétrique élémentaire $\Sigma\, x_1 \cdot \cdot \cdot x_i$ [3] c'est donc un élément de $I_G = H^*(B_{\boldsymbol{U}(n)}, Z)$ et de plus I_G est identique à l'algèbre des polynômes en les C_{2i}; comparant cela avec les résultats du No. 2, on voit que $H^*(\boldsymbol{U}(n), Z)$ contient des éléments bien déterminés h_i, $(1 \leq i \leq n)$, de degré $2i - 1$, tels que $\tau(h_i) = C_{2i} \bmod (C_2, \cdot \cdot \cdot, C_{2i-2})$, τ étant la transgression dans $E_{\boldsymbol{U}(n)}$; en outre $H^*(\boldsymbol{U}(n), Z)$ est l'algèbre extérieure engendrée par les h_i.

Il s'impose de comparer les résultats précédents avec ceux qu'obtient S. S. Chern dans [6]. Au moyen des grassmanniennes complexes et des symboles de Schubert, Chern y définit des éléments $c_{2i} \in H^{2i}(B_{\boldsymbol{U}(n)}, Z)$ et montre que $H^*(B_{\boldsymbol{U}(n)}, Z)$ est l'algèbre des polynômes admettant les c_{2i}, $(1 \leq i \leq n)$, comme générateurs; ainsi les classes C_{2i} ont les mêmes propriétés que les classes de Chern c_{2i}; en fait on a:

Proposition 4.1. *Les classes* $C_{2i} = \Sigma\, x_1 \cdot \cdot \cdot x_i$ *coïncident avec les classes de Chern* c_{2i} $(i = 1, \cdot \cdot \cdot, n)$.

Pour éviter toute confusion, nous n'identifierons pas dans cette démonstration $I_{\boldsymbol{U}(n)}$ et $H^*(B_{\boldsymbol{U}(n)}, Z)$. Nous devons donc montrer que *l'homomorphisme* $\rho^*(\boldsymbol{T}, \boldsymbol{U}(n)) : H^*(B_{\boldsymbol{U}(n)}, Z) \to H^*(B_{\boldsymbol{T}}, Z)$ *applique* c_{2i} *sur* C_{2i}.

Nous établirons la Prop. 4.1 par récurrence sur n en utilisant la *formule de dualité* des classes de Chern (4.3), (*voir* [7], [19]). Pour $n = 1$, $\boldsymbol{U}(1) = \boldsymbol{T}$ et la proposition est évidente car C_2 et c_2 sont toutes deux images par transgression de la classe $\xi = d\zeta$ de $H^1(\boldsymbol{T}, Z)$. Nous supposons maintenant la proposition démontrée pour $\boldsymbol{U}(m)$ si $m < n$; comme nous aurons à considérer plusieurs valeurs de n, il sera commode de distinguer par un indice n, que nous placerons en haut à gauche, ce qui est relatif à $\boldsymbol{U}(n)$; ainsi nous parlerons de ${}^n\xi_i$, ${}^nx_i$, ${}^nC_{2i}$, ${}^nc_{2i}$, etc.

Soient $p, q > 0$ tels que $p + q = n$, f l'inclusion canonique de $\boldsymbol{U}(p) \times \boldsymbol{U}(q)$ dans $\boldsymbol{U}(n)$, $\boldsymbol{T}^p$, $\boldsymbol{T}^q$ et $\boldsymbol{T}^n = \boldsymbol{T}^p \times \boldsymbol{T}^q$ des tores maximaux de $\boldsymbol{U}(p)$, $\boldsymbol{U}(q)$ et

[3] Dans tout ce travail, nous notons un polynôme symétrique par son terme initial précédé du signe Σ.

$\boldsymbol{U}(n)$; la restriction g de f à $\boldsymbol{T}^p \times \boldsymbol{T}^q$ est donc l'identité. Le diagramme commutatif

$$\begin{array}{ccc} \boldsymbol{T}^p \times \boldsymbol{T}^q & \xrightarrow{g} & \boldsymbol{T}^n \\ \downarrow & & \downarrow \\ \boldsymbol{U}(p) \times \boldsymbol{U}(q) & \xrightarrow{f} & \boldsymbol{U}(n) \end{array}$$

où les flèches sont des inclusions, donne lieu au diagramme commutatif

$$\begin{array}{ccc} H^*(B_{\boldsymbol{T}^p}, Z) \otimes H^*(B_{\boldsymbol{T}^q}, Z) & \xleftarrow{\alpha} & H^*(B_{\boldsymbol{T}^n}, Z) \\ \uparrow \beta & & \uparrow \delta \\ H^*(B_{\boldsymbol{U}(p)}, Z) \otimes H^*(B_{\boldsymbol{U}(q)}, Z) & \xleftarrow{\gamma} & H^*(B_{\boldsymbol{U}(n)}, Z) \end{array}$$

où, dans les notations du No. 1 :

$$\alpha = \rho^*(g) = \rho^*(\boldsymbol{T}^p \times \boldsymbol{T}^q, \boldsymbol{T}^n), \qquad \beta = \rho^*(\boldsymbol{T}^p \times \boldsymbol{T}^q, \boldsymbol{U}(p) \times \boldsymbol{U}(q)),$$
$$\gamma = \rho^*(\boldsymbol{U}(p) \times \boldsymbol{U}(q), \boldsymbol{U}(n)) = \rho^*(f) \text{ et enfin } \delta = \rho^*(\boldsymbol{T}^n, \boldsymbol{U}(n)).$$

Si l'on prend dans $H^1(\boldsymbol{T}^p, Z)$, $H^1(\boldsymbol{T}^q, Z$ et $H^1(\boldsymbol{T}^n, Z)$ les bases $({}^p\xi_j)$, $({}^q\xi_k)$, $({}^n\xi_i)$ indiquées au début de ce No, il est clair que g^* est définie par

$$g^*({}^n\xi_i) = {}^p\xi_i \quad (i \leq p), \qquad g^*({}^n\xi_{p+i}) = {}^q\xi_i, \quad (1 \leq i \leq q),$$

par conséquent, α est défini par

$$\alpha({}^n x_i) = {}^p x_i \otimes 1 \quad (1 \leq i \leq p) ; \qquad \alpha({}^n x_{p+j}) = 1 \otimes {}^q x_j \quad (1 \leq j \leq q),$$

et il en résulte évidemment que

$$\alpha({}^nC_{2i}) = \Sigma_{j+k=i}\ {}^pC_{2j} \otimes {}^qC_{2k} \tag{4.2}$$

en posant bien entendu ${}^pC_{2j} = 0$ si $j > p$, ${}^qC_{2k} = 0$ si $k > q$; en faisant une convention analogue pour les classes de Chern, on peut écrire la formule de dualité :

$$\gamma({}^nc_{2i}) = \Sigma_{j+k=i}\ {}^pc_{2j} \otimes {}^qc_{2k} \tag{4.3}$$

mais l'homomorphisme β est le produit tensoriel des homomorphismes $\rho^*(\boldsymbol{T}^p, \boldsymbol{U}(p))$ et $\rho^*(\boldsymbol{T}^q, \boldsymbol{U}(q))$, donc, vu l'hypothèse d'induction, on obtient :

$$\beta \circ \gamma({}^nc_{2i}) = \Sigma_{j+k=i}\ {}^pC_{2j} \otimes {}^qC_{2k} \tag{4.4}$$

d'où, compte tenu de (4.2) et de $\beta \circ \gamma = \alpha \circ \delta$,

$$\alpha \circ \delta({}^nc_{2i}) = \alpha({}^nC_{2i}), \qquad (1 \leq i \leq n),$$

et finalement $\delta({}^nc_{2i}) = {}^nC_{2i}$ puisque α est biunivoque,

Remarques. (1) Sans utiliser la formule de dualité on peut montrer aisément que ${}^nc_{2i} = \epsilon_i C_{2i}$, où $\epsilon_i = \pm 1$. Pour prouver la Proposition 4. 1, il ne reste donc plus qu'à établir les égalités $\epsilon_i = 1$ $(1 \leq i \leq n)$, ce qui peut se faire en se servant des valeurs des classes de Chern de la structure tangente de l'espace projectif complexe et du calcul des puissances de Steenrod des classes ${}^nC_{2i}$ qui sera fait au No. 11. Malheureusement cette méthode, au reste peu naturelle, conduit à des calculs assez compliqués.

Il serait intéressant de trouver une démonstration de la Proposition 4. 1 qui soit simple et indépendante de la dualité; peut être est-ce possible en utilisant l'expression des classes de Chern comme formes différentielles? On déduirait alors la formule de dualité directement de l'identité évidente (4. 2), comme cela est fait dans [3] pour les classes de Stiefel-Whitney réduites mod 2.

5. Autres groupes classiques. Nous allons passer brièvement en revue les autres groupes classiques, renvoyant à [2] pour plus de détails.[4]

Examinons tout d'abord les groupes *orthogonaux*. Le groupe $\boldsymbol{SO}(2n+1)$ est de rang n, et son groupe de Weyl est le groupe des permutations et changements de signes des x_i. Il en résulte donc que $H^*(B_{\boldsymbol{SO}(2n+1)}, Z_p)$ est l'algèbre des polynômes ayant comme générateurs les n éléments $P_{4i} = \Sigma x_1^2 \cdot \cdots x_i^2$, $(1 \leq i \leq n)$, lorsque p est impair (cette restriction étant due, rappelons-le, au fait que $\boldsymbol{SO}(n)$ a de la 2-torsion pour $m \geq 3$).

Le groupe $\boldsymbol{SO}(2n)$ est aussi de rang n, et son groupe de Weyl est engendré par les permutations et les changements de signes *en nombre pair* des x_i. Il en résulte aisément que pour p premier impair $H^*(B_{\boldsymbol{SO}(2n)}, Z_p)$ est l'algèbre des polynômes admettant comme générateurs les P_{4i} $(1 \leq i \leq n-1)$, et $W_{2n} = x_1 \cdot \cdots x_n$.

Proposition 5. 1. *P_{4i} coïncide avec la classe de Pontrjagin de dimension $4i$, réduite mod p; W_{2n} coïncide avec la classe de Stiefel-Whitney de dimension $2n$, réduite mod p.*

Nous noterons p_{4i} et w_{2n} les classes de Pontrjagin et de Stiefel-Whitney respectivement.

Soit d'abord f l'inclusion de $U(n)$ dans $SO(2n)$; ces deux groupes étant de même rang, $\rho^*(f) = \rho^*(\boldsymbol{U}(n), \boldsymbol{SO}(2n))$ est biunivoque et, vu les identifications faites plus haut, se ramène au plongement des polynômes en les P_{4i}

[4] Pour tout ce qui concerne le groupe de Weyl, *voir* par exemple E. Stiefel, *Comm. Math. Helv.*, tome 14 (1941-42), pp. 350-380.

et W_{2n} dans l'algèbre de tous les polynômes symétriques. Il en résulte visiblement:

$$\rho^*(f)(W_{2n}) = C_{2n}. \tag{5.2}$$

Mais il est classique ([13], 41. 8) que, étant donné un espace fibré de groupe structural $\boldsymbol{U}(n)$ et de classes de Chern c_{2i}, on obtient en étendant le groupe structural à $\boldsymbol{SO}(2n)$ un espace fibré dont la classe de Stiefel-Whitney w_{2n} est égale à c_{2n}; cela signifie que $\rho^*(f)(w_{2n}) = c_{2n}$ et, comme $C_{2n} = c_{2n}$, l'égalité (5. 1) donne bien $W_{2n} = w_{2n}$, puisque $\rho^*(f)$ est biunivoque.

On pourrait raisonner de la même façon pour les classes de Pontrjagin, à l'aide du No. 3 de la Note [18] de Wu, mais il est plus commode de procéder différemment, en partant du plongement canonique de $\boldsymbol{SO}(n)$ dans $\boldsymbol{U}(n)$, qui définit un homomorphisme

$$\sigma = \rho^*(\boldsymbol{SO}(n), \boldsymbol{U}(n)) : H^*(B_{\boldsymbol{U}(n)}, Z_p) \to H^*(B_{\boldsymbol{SO}(n)}, Z_p)$$

déterminé explicitement dans [2], § 31; on a

$$\begin{aligned} &\sigma(C_{2i}) = 0 \text{ si } i \text{ est impair} \\ &\sigma(C_{4k}) = (-1)^k P_{4k}. \end{aligned} \tag{5.3}$$

D'autre part, on a d'après Wu ([20], p. 9):

$$\sigma(c_{4k}) = (-1)^k p_{4k},$$

ce qui, joint à $C_{4k} = c_{4k}$, donne $P_{4k} = p_{4k}$.

Remarque. De même que pour la Proposition 4. 1, il y aurait intérêt à avoir une démonstration de la Proposition 5. 1 indépendante des résultats de Wu, car on obtiendrait alors ces derniers de façon simple, par des calculs sur les polynômes symétriques et des changements de variables.

Le cas du groupe unitaire symplectique $\boldsymbol{Sp}(n)$ est tout à fait analogue à celui du groupe $\boldsymbol{SO}(2n+1)$, car ces deux groupes ont même rang et des groupes de Weyl isomorphes; la différence essentielle est que l'on peut raisonner directement avec des coefficients entiers puisque $\boldsymbol{Sp}(n)$ n'a pas de torsion.

On trouve alors que $H^*(B_{\boldsymbol{Sp}(n)}, Z)$ est identique à l'algèbre des polynômes ayant pour générateurs des classes K_{4i}, $(1 \leq i \leq n)$, définies par: $K_{4i} = \Sigma\, x_1^2 \cdots x_i^2$.

Le plongement canonique de $\boldsymbol{Sp}(n)$ dans $\boldsymbol{U}(2n)$ définit un homomorphisme

$$\nu = \rho^*(\boldsymbol{Sp}(n)), \boldsymbol{U}(2n)) \quad : \quad H^*(B_{\boldsymbol{U}(2n)}, Z) \to H^*(B_{\boldsymbol{Sp}(n)}, Z)$$

donné par les formules:

$$\nu(C_{2i}) = 0 \text{ si } i \text{ est impair}$$

(5.4)

$$\nu(C_{4k}) = (-1)^k K_{4k}.$$

Note. Les formules (5.2), (5.3), (5.4) permettent de déterminer les homomorphismes

$$H^*(\boldsymbol{SO}(2n), Z_p) \to H^*(\boldsymbol{U}(n), Z_p) \to H^*(\boldsymbol{SO}(n), Z_p)$$

et

$$H^*(\boldsymbol{U}(2n), Z) \to H^*(\boldsymbol{Sp}(n), Z)$$

induits par les plongements canoniques ([2], Proposition 21.3 et § 31). Ainsi, si l'on désigne par t_k l'élément de $H^{4k-1}(\boldsymbol{SO}(n), Z_p)$ dont l'image par transgression est P_{4k}, par u_n celui dont l'image est W_n (n pair), par v_k l'élément de $H^{4k-1}(\boldsymbol{Sp}(n), Z)$ dont l'image est K_{4k}, on voit que $(-1)^k t_k$ et $(-1)^k v_k$ sont restrictions des classes

$$h_{2k} \in H^{4k-1}(\boldsymbol{U}(n), Z_p), \text{ resp. } h_{2k} \in H^{4k-1}(\boldsymbol{U}(n), Z).$$

Par conséquent:

5.5. *A l'exception de la classe u_n,* (d'ailleurs exceptionnelle à plus d'un titre), *les générateurs des algèbres de cohomologie des groupes classiques sont restrictions de générateurs de l'algèbre de cohomologie du groupe unitaire* (*pour $\boldsymbol{SO}(n)$, on suppose $p \neq 2$*).

II. Les puissances réduites de Steenrod.

6. Les puissances réduites. Dans les Nos. 6 et 7, p désignera un nombre premier fixé, K un polyèdre fini, L un sous-polyèdre de K. On notera $H^q(K, L; Z_p)$ ou $H^q(K, L)$ lorsque cela ne prêtera pas à confusion, les groupes de cohomologie de K modulo L, à coefficients dans Z_p.

Les puissances réduites sont des homomorphismes

$$P_i^p : H^q(K, L; Z_p) \to H^{pq-i}(K, L; Z_p)$$

définis pour tout p premier, tout $i \geq 0$, tout $q \geq 0$, et tout couple de polyèdres K et L, L étant un sous-polyèdre de K.

A la place de la notation P_i^p, qui est adoptée dans [15], on trouve fréquemment ([4], [17], [20]) la notation St_p^i qui désigne $P^{p}{}_{pq-q-i}$; ainsi l'homomorphisme St_p^i applique $H^q(K, L)$ dans $H^{q+i}(K, L)$ et élève donc le degré de i unités.

Dans cet article, nous utiliserons une troisième notation, qui figure à la fin de [15b]. Rappelons les raisons de ce changement: D'après un théorème de Thom, ([16], [15b]), on a $St_p^i = 0$ si $i \not\equiv 0 \bmod 2\ (p-1)$ et il est connu que St_p^{2i+1} se ramène immédiatement à St_p^{2i}; ainsi, seules les opérations $St_p^{2k(p-1)}$ sont réellement importantes, et il est naturel de les désigner par un symbole plus simple; d'autre part les propriétés des St_p^i sont compliquées par la présence de facteurs numériques; si on cherche à les faire disparaître, en modifiant convenablement la définition des puissances réduites, on est finalement conduit à la notation suivante, qui est celle que nous adopterons dans toute la suite:

On désigne par $\mathcal{P}_p^k$ *l'homomorphisme de* $H^q(K, L; Z_p)$ *dans*

$$H^{q+2k(p-1)}(K, L; Z_p)$$

qui est égal à $\lambda(p, q, k) \cdot St_p^{2k(p-1)}$, *le coefficient* $\lambda(p, q, k)$ *étant un élément de* Z_p *défini comme suit:*

Si $p = 2$, $\lambda(p, q, k) = 1$

Si $p = 2h + 1$, $\lambda(p, q, k) = (-1)^{hr(r-1)/2}(h!)^{-r}$ *avec* $r = q - 2k$.

7. Formulaire. Nous rappelons ici les propriétés des puissances réduites $\mathcal{P}_p^k$, (*voir* [15b]); dans la suite de ce travail, nous n'utiliserons que ces formules, et jamais la définition explicite des $\mathcal{P}_p^k$, (ce qui n'est d'ailleurs pas surprenant, puisque Thom [17] a montré que les formules en question caractérisent complètement les $\mathcal{P}_p^k$).

7.1. $\mathcal{P}_p^k$: $H^q(K, L; Z_p) \to H^{q+2k(p-1)}(K, L; Z_p)$ *est un homomorphisme défini quels que soient* $q \geq 0$, $k \geq 0$ *et le couple* (K, L).

7.2. *Soit* $f: (K, L) \to (K', L')$ *une application continue. Si* f^* *est l'homomorphisme de* $H^*(K', L')$ *dans* $H^*(K, L)$ *induit par* f *on a* $\mathcal{P}_p^k \circ f^* = f^* \circ \mathcal{P}_p^k$.

7.3. *Lorsque* $p = 2$, *on a* $\mathcal{P}_p^k = Sq^{2k}$ (" *i-carré* " *de Steenrod*).

7.4. $\mathcal{P}_p^0$ *est l'application identique de* $H^*(K, L)$ *sur lui-même.*

7.5. $\mathcal{P}_p^k: H^q(K, L) \to H^{q+2k(p-1)}(K, L)$ *est nul lorsque* $q < 2k$, *coïncide avec l'élévation à la p-ième puissance lorsque* $q = 2k$.

7.6. *Si* δ *désigne l'homomorphisme cobord qui applique* $H^q(L)$ *dans* $H^{q+1}(K, L)$, *on a* $\mathcal{P}_p^k \circ \delta = \delta \circ \mathcal{P}_p^k$.

7.7. *Soient x et y deux éléments de $H^*(K, L)$, $x \cdot y$ leur cup-produit. Lorsque $p \neq 2$, on a:*

$$\mathcal{P}_p^k(x \cdot y) = \Sigma_{i+j=k} \mathcal{P}_p^i(x) \cdot \mathcal{P}_p^j(y).$$

(Cela montre en particulier que $\mathcal{P}_p^1$ est une *dérivation.*)

7.8. $Sq^k(x \cdot y) = \Sigma_{i+j=k}\ Sq^i(x) \cdot Sq^j(y)$.

(*Ainsi, la formule 7.7 est valable pour $p = 2$ quand Sq^1 est nul pour tout élément de $H^*(K, L)$.*)

Remarquons enfin que la transgression dans un espace fibré, dont nous avons rappelé la définition au No. 2, est un produit $q^{*-1} \cdot \delta$, où q^* et δ commutent avec $\mathcal{P}_p^k$, vu 7.2 et 7.6; par conséquent:

7.9. *Si τ désigne la transgression dans un espace fibré E on a:*

$$\mathcal{P}_p^k \circ \tau = \tau \circ \mathcal{P}_p^k.$$

De façon plus précise, $\mathcal{P}_p^k$ applique $T^s(F)$ dans $T^{s+2k(p-1)}(F)$, et $L^{s+1}(B)$ dans $L^{s+1+2k(p-1)}(B)$, et on a un diagramme commutaif

$$\begin{array}{ccc} T^s(F) & \xrightarrow{\mathcal{P}_p^k} & T^{s+2k(p-1)}(F) \\ \downarrow \tau & & \downarrow \tau \\ H^{s+1}(B)/L^{s+1}(B) & \xrightarrow{\mathcal{P}_p^k} & H^{s+1+2k(p-1)}(B)/L^{s+1+2k(p-1)}(B) \end{array}$$

Note. Les puissances réduites $\mathcal{P}_p^k$ sont définies dans [15] pour les couples (K, L) de *polyèdres finis*; mais, comme nous l'a signalé N. E. Steenrod, elles sont définissables dans des théories cohomologiques plus vastes et notament dans la *théorie de Čech* (par passage à la limite à partir des polyèdres) et dans la *théorie singulière* (car il existe une formule simpliciale universelle, analogue à celle des i-produits, qui fait passer d'un cocycle représentant la classe de cohomologie x à un cocycle représentant $\mathcal{P}_p^k(x)$). Bien entendu, le formulaire précédent est encore valable dans ces deux cas.

8. Les puissances réduites dans les espaces projectifs. Nous allons montrer maintenant comment les formules du No. 7 permettent de déterminer les opérations $\mathcal{P}_p^k$ dans quelques cas simples. Les résultats obtenus nous serviront du reste dans la troisième partie de ce travail.

PROPOSITION 8.1. *Soit u une classe de cohomologie de dimension 2. On a, si $p \neq 2$, $\mathcal{P}_p^k(u^n) = \binom{n}{k} u^{n+k(p-1)}$ (on convient que $\binom{n}{k} = 0$ si $k > n$); si $p = 2$ la formule précédente reste valable lorsque $Sq^1 u = 0$.*

On raisonne par récurrence sur n. Supposons tout d'abord $p \neq 2$; en appliquant 7.7 on obtient:

$$\mathcal{P}_p^k(u^n) = \Sigma_{i+j=k} \mathcal{P}_p^i(u) \cdot \mathcal{P}_p^j(u^{n-1})$$

et d'après l'hypothèse de récurrence, la somme du 2ème membre se réduit à

$$\binom{n-1}{k-1} u^{n+k(p-1)} + \binom{n-1}{k} u^{n+k(p-1)} = \binom{n}{k} u^{n+k(p-1)}.$$

Pour $p = 2$, on doit appliquer la formule 7.8 et on trouve un terme supplémentaire, égal à $Sq^1 u \cdot Sq^{2k-1}(u^{n-1})$, qui est nul si l'on suppose que $Sq^1 u = 0$, et le reste du calcul vaut sans changement.

COROLLAIRE 8.2. *Soient $\boldsymbol{X} = \boldsymbol{P}_m(C)$ l'espace projectif complexe de dimension complexe m, u un élément de $H^2(\boldsymbol{X}, Z_p)$. On a:*

$$\mathcal{P}_p^k(u^n) = \binom{n}{k} u^{n+k(p-1)}.$$

La seule chose à vérifier est que $Sq^1 u = 0$, ce qui résulte de la nullité de $H^3(\boldsymbol{X}, Z_2)$.

COROLLAIRE 8.3. *Soit $\boldsymbol{Y} = \boldsymbol{P}_m(K)$ l'espace projectif quaternionien de dimension quaternionienne m. Alors $H^4(\boldsymbol{Y}, Z_p)$ contient un élément $v \neq 0$ tel que:*

$$\mathcal{P}_p^k(v^n) = 0 \text{ si } p = 2 \text{ et si } k \text{ est impair,}$$

$$\mathcal{P}_p^k(v^n) = \binom{2n}{k} v^{n+k(p-1)/2} \text{ sinon.}$$

Par définition même, $\boldsymbol{P}_m(K)$ est la base de $\boldsymbol{S}_{4m+3}$, fibrée par le groupe $\boldsymbol{Sp}(1)$ des quaternions de norme 1, qui est homéomorphe à $\boldsymbol{S}_3$; de même le quotient de $\boldsymbol{S}_{4m+3}$ par un sous-groupe $\boldsymbol{S}_1$ de $\boldsymbol{Sp}(1)$ est l'espace $\boldsymbol{P}_{2m+1}(C)$. Ce dernier est donc fibré de fibre $\boldsymbol{S}_3/\boldsymbol{S}_1 = \boldsymbol{S}_2$ et de base $\boldsymbol{P}_m(K)$; soit ψ la projection qui définit cette fibration. On voit immédiatement (soit par un raisonnement géométrique, soit en examinant la suite spectrale de cette fibration), que ψ^* est un isomorphisme de $H^*(\boldsymbol{Y})$ sur la sous-algèbre de $H^*(\boldsymbol{P}_{2m+1}(C))$ engendrée par u^2, u étant un générateur de $H^*(\boldsymbol{P}_{2m+1}(C))$. Soit alors v la réduction mod p de l'élément $\hat{v} \in H^4(\boldsymbol{Y}, Z)$ tel que $\psi^*(\hat{v}) = u^2$.

Pour $p = 2$, k impair, $\mathcal{P}_p^k(v^n)$ est de dimension congrue à 2 mod 4, et est donc nul; sinon on a

$$\psi^*(\mathcal{P}_p^k(v^n)) = \mathcal{P}_p^k(u^{2n}) = \binom{2n}{k} u^{2n+k(p-1)} = \phi^*\left(\binom{2n}{k} v^{n+k(p-1)/2}\right),$$

d'où le corollaire, puisque ψ^* est biunivoque.

Remarque. Il est naturel de se demander si, de même que dans le

Corollaire 8.2, la formule du corollaire 8.3 vaut pour *tout* élément de $H^4(Y)$. Un tel élément étant de la forme λv, $(\lambda \in Z_p)$, on est amené à voir si l'on a $\lambda^{k(p-1)/2} \equiv 1 \bmod p$; si k est pair, c'est bien le cas quel que soit $\lambda \not\equiv 0$, mais si k est impair, il faut et il suffit que λ soit reste quadratique mod p.

Proposition 8.4. *Soit X un espace dont l'algèbre de cohomologie $H^*(X, Z_p)$ est engendrée par des éléments de dimension 2. Si $(\mathcal{P}_p^1)^k$ désigne l'opération $\mathcal{P}_p^1$ itérée k fois, on a $(\mathcal{P}_p^1)^k(x) = k!\mathcal{P}_p^k(x)$ pour tout $x \in H^*(X, Z_p)$.*

Cette égalité résulte immédiatement de la Proposition 8.1 pour x de dimension deux; il nous reste donc simplement à montrer que si elle est vraie pour x et y, elle l'est encore pour $x \cdot y$; or, $\mathcal{P}_p^1$ étant une dérivation d'après 7.7, on peut lui appliquer la formule de Leibnitz donnant la dérivée k-ième d'un produit, et l'on obtient ainsi:

$$(\mathcal{P}_p^1)^k(x \cdot y) = \Sigma_{i+j=k} \binom{k}{i} (\mathcal{P}_p^1)^i(x) \cdot (\mathcal{P}_p^1)^j(y).$$

D'après l'hypothèse faite sur x et y, le second membre est égal à

$$\Sigma_{i+j=k}\, i!j!\binom{k}{i} \cdot \mathcal{P}_p^i(x) \cdot \mathcal{P}_p^j(y) = k!\, \Sigma_{i+j=k} \mathcal{P}_p^i(x) \cdot \mathcal{P}_p^j(y),$$

donc à $k!\mathcal{P}_p^k(x \cdot y)$, d'après 7.7.

Corollaire. *L'opération $\mathcal{P}_p^1$, itérée p fois, est nulle.*

Remarque. Si $p \neq 2$ la formule $(\mathcal{P}_p^1)^k = k!\mathcal{P}_p^k$ est très probablement valable sans hypothèse restrictive sur X; elle l'est en tout cas lorsque $X = G$ et $X = B_G$, G étant un groupe de Lie vérifiant les conditions 3.2, car $H^*(B_G)$ est alors isomorphe à une sous-algèbre de $H^*(B_T)$, laquelle vérifie les hypothèses de la Proposition 8.4, et on passe de là à $H^*(G)$ par transgression. Cette formule est par contre inexacte pour $p = 2$, comme le montre l'exemple $Sq^2 \circ Sq^2 = Sq^3 \circ Sq^1$.

9. Applications aux groupes d'homotopie des sphères. N. E. Steenrod, (*Reduced powers of cohomology classes*, Cours professé au Collège de France, Mai 1951), a montré comment on peut utiliser les puissances réduites pour étudier les groupes $\pi_i(S_n)$. Rappelons sa méthode:

Soient p un nombre premier, k un entier, f une application continue de S_i dans S_n, avec $i = n + 2k(p-1) - 1$. Désignons par X le complexe cellulaire obtenu en adjoignant à S_n une boule de dimension $i+1$ par l'application f de sa frontière dans S_n; on a:

$$H^0(X, Z) = H^n(X, Z) = H^{n+2k(p-1)}(X, Z) = Z$$

et les autres groupes de cohomologie de $\boldsymbol{X}$ sont nuls; soit encore s, (resp. t), la réduction mod p du générateur canonique de $H^n(\boldsymbol{X}, Z)$, (resp. de $H^{n+2k(p-1)}(\boldsymbol{X}, Z)$). On a $\mathcal{P}_p{}^k(s) = \lambda_f t$, $(\lambda_f \in Z_p)$, et il est clair que λ_f ne dépend que de la classe d'homotopie de f, et que l'application $f \to \lambda_f$ définit un homomorphisme de $\pi_{n+2k(p-1)-1}(\boldsymbol{S}_n)$ dans Z_p que nous noterons $\zeta_p{}^{n,k}$.

Si E désigne la suspension de Freudenthal, on a

$$\zeta_p{}^{n+1,k} \circ E = \zeta_p{}^{n,k}, \tag{9.1}$$

(cela résulte du fait que $\mathcal{P}_p{}^k$ commute avec δ).

Exemples.

1) $p = 2$. On déduit de l'existence d'applications dont l'invariant de Hopf est 1 que $\zeta_2{}^{n,1}$, $\zeta_2{}^{n,2}$ et $\zeta_2{}^{n,4}$ sont des homomorphismes de $\pi_{n+1}(\boldsymbol{S}_n)$, de $\pi_{n+3}(\boldsymbol{S}_n)$ et de $\pi_{n+7}(\boldsymbol{S}_n)$ *sur* Z_2 (pour $n \geq 2$, $n \geq 4$ et $n \geq 8$ respectivement).

2) $p = 3$. Il est classique que l'espace $\boldsymbol{X}$ obtenu par le procédé décrit plus haut à partir de l'application de Hopf $f : \boldsymbol{S}_7 \to \boldsymbol{S}_4$ est le *plan projectif quaternionien* $\boldsymbol{P}_2(K)$. Or, d'après 8.3, on a $\mathcal{P}_3{}^1(v) \not\equiv 0 \bmod 3$ si v est un élément non nul de $H^4(\boldsymbol{P}_2(K), Z_3)$; par conséquent $\zeta_3{}^{4,1}(f) \neq 0$, ce qui, compte tenu de 9.1, montre que $\zeta_3{}^{n,1}$ est un homomorphisme de $\pi_{n+3}(\boldsymbol{S}_n)$ *sur* Z_3 pour $n \geq 4$.

La suspension itérée E^2 étant un isomorphisme de $\pi_6(\boldsymbol{S}_3)$ sur un sous-groupe d'indice 2 de $\pi_8(\boldsymbol{S}_5)$, (d'après des résultats classiques de Freudenthal et de G. W. Whitehead), la formule 9.1 montre que $\zeta_3{}^{3,1}$ applique $\pi_6(\boldsymbol{S}_3)$ *sur* Z_3, résultat obtenu d'une autre manière par Steenrod, et que nous préciserons au No. 19 en donnant explicitement un élément de $\pi_6(\boldsymbol{S}_3)$ dont l'image par $\zeta_3{}^{3,1}$ est $\neq 0$.

Ainsi, pour $p = 2, 3$, l'homomorphisme $\zeta_p{}^{n,1}$ applique $\pi_{n+2p-3}(\boldsymbol{S}_n)$ sur Z_p lorsque $n \geq 3$; nous allons voir que ce fait est général; plus précisément:

PROPOSITION 9.2. *L'homomorphisme* $\zeta_p{}^{n,1} : \pi_{n+2p-3}(\boldsymbol{S}_n) \to Z_p$ *est un isomorphisme du p-composant de* $\pi_{n+2p-3}(\boldsymbol{S}_n)$ *sur* Z_p *lorsque* $n \geq 3$.

D'après [13], Chap. IV, Prop. 3 et 4, la suspension de Freudenthal applique isomorphiquement le p-composant de $\pi_{n+2p-3}(\boldsymbol{S}_n)$ sur celui de $\pi_{n+2p-2}(\boldsymbol{S}_{n+1})$ lorsque $n \geq 3$. Vu 9.1, il suffit donc de démontrer 9.2 pour $n = 3$; comme on sait que le p-composant de $\pi_{2p}(\boldsymbol{S}_3)$ est Z_p, (ibid. Proposition 7), on est finalement ramené à prouver que, si $f : \boldsymbol{S}_{2p} \to \boldsymbol{S}_3$ est une application essentielle définissant un élément d'ordre p de $\pi_{2p}(\boldsymbol{S}_3)$, on a $\lambda_f \neq 0$, avec les notations introduites au début de ce paragraphe.

Or, soit X l'espace obtenu en adjoignant à S_3 une cellule de dimension $2p+1$ à l'aide de f, et écrivons la suite exacte d'homotopie de (X, S_3) :

$$\cdots \to \pi_i(S_3) \to \pi_i(X) \to \pi_i(X, S_3) \xrightarrow{d} \pi_{i-1}(S_3) \to \pi_{i-1}(X) \to \cdots$$

On a visiblement $\pi_i(X, S_3) = 0$ lorsque $i < 2p+1$ et $\pi_{2p+1}(X, S_3) = Z$; en outre, l'image de $d : \pi_{2p+1}(X, S_3) \to \pi_{2p}(S_3)$ est le sous-groupe engendré par la classe de f, et est donc isomorphe à Z_p. Il suit de là :

$$\pi_i(X) \approx \pi_i(S_3) \text{ si } i < 2p, \qquad \pi_{2p}(X) \approx \pi_{2p}(S_3)/Z_p.$$

Soit alors $Y = (X, 4)$ l'espace obtenu à partir de X en tuant $\pi_3(X) = Z$, (au sens de [5], voir aussi [13], Chap. III). Par définition de Y, on a $\pi_i(Y) = 0$ pour $i \leq 3$, et $\pi_i(Y) = \pi_i(X)$ pour $i \geq 4$, ce qui, joint aux formules précédentes, montre que $\pi_i(Y)$, $(i \leq 2p)$, est un groupe fini dont le p-composant est nul; il en résulte que $H^i(Y, Z_p) = 0$ lorsque $0 < i \leq 2p$, ([13], Chap. III, Théor. 1).

Mais d'autre part Y est un espace fibré de base X et dont la fibre est un $K(Z, 2)$, au sens d'Eilenberg-MacLane. L'algèbre $H^*(Z, 2)$ est, comme on sait, une algèbre de polynômes engendrée par un élément de dimension deux, soit r. La transgression τ dans l'espace fibré Y transforme r en un élément de $H^3(X)$, qui est nécessairement de la forme λs, $(\lambda \in Z_p)$, s étant le générateur introduit plus haut. L'élément r étant transgressif, il en est de même de $r^p = \mathcal{P}_p^1(r)$, d'après 7. 9, et l'on a : $\tau(r^p) = \mathcal{P}_p^1(\tau(r)) = \lambda \mathcal{P}_p^1(s)$.

Si $\tau(r^p)$ étant nul, r^p définirait un élément non nul de $H^{2p}(Y, Z_p)$ ce qui est impossible, on l'a vu; on doit donc forcément avoir $\mathcal{P}_p^1(s) \neq 0$, ce qui signifie justement que $\lambda_f \neq 0$.

On remarquera que la démonstration précédente, à la différence de celles relatives à $p = 2$ et à $p = 3$, ne fournit aucun élément *explicite* de $\pi_{n+2p-3}(S_n)$ dont l'image par $\zeta_p^{n,1}$ soit non nulle.

III. Les puissances réduites dans la cohomologie des groupes de Lie et de leurs espaces classifiants.

10. Méthode générale. Nous revenons maintenant sur la méthode de calcul des puissances réduites dans $H^*(G, Z_p)$ et $H^*(B_G, Z_p)$ déjà brièvement décrite dans l'introduction.

p étant un nombre premier arbitraire, mais fixé, nous supposerons *dans toute la suite* que G vérifie les conditions 3. 2, autrement dit que G et son quotient G/T par un tore maximal sont sans p-torsion; comme nous l'avons

rappelé au No. 3, cela a lieu pour tout groupe classique et tout p, à l'exception du cas $G = \boldsymbol{SO}(n)$, $p = 2$; nous n'obtiendrons donc pas ici les Sq^i dans $H^*(\boldsymbol{SO}(n), Z_2)$ et $H^*(B_{\boldsymbol{SO}(n)}, Z_2)$, pour lesquels nous renvoyons à [3], où ils sont traités par une méthode analogue, utilisant des sous-groupes abéliens maximaux de type $(2, 2, \cdots, 2)$ au lieu de tores maximaux.

G vérifiant 3. 2, l'algèbre $H^*(B_G, Z_p)$ est une algèbre de polynômes à l générateurs de dimensions paires, soient $y_1, \cdots, y_l$, qui s'identifie canoniquement à une sous-algèbre de $H^*(B_T, Z_p)$; or cette dernière est une algèbre de polynômes à l générateurs $x_1, \cdots, x_l$ de dimension deux et les p-puissances réduites y sont déterminées par la Proposition 8. 1 et la formule 7. 7 ; cela résout donc la question pour $H^*(B_G, Z_p)$.

On passe de là à $H^*(G, Z_p)$ par transgression en utilisant les résultats du No. 2. Soit x_i l'élément universellement transgressif de $H^*(G, Z_p)$ tel que $\tau(x_i) = y_i \bmod D$, $(1 \leq i \leq l)$; d'après 7. 9, $\mathcal{P}_p^k(x_i)$ est aussi universellement transgressif et de plus, compte tenu de 2. 3 :

$$\mathcal{P}_p^k(x_i) = \mathcal{P}_p^k(\tau x_i) = \mathcal{P}_p^k(y_i) \bmod D\tau ;$$

mais les puissances réduites dans $H^*(B_G, Z_p)$ sont déjà connues ; on sait donc exprimer $\mathcal{P}_p^k(y_i)$ comme polynôme en les y_j ; désignons par $\Sigma \lambda_j y_j$ sa partie homogène de degré 1. On a donc $\mathcal{P}_p^k(y_i) = \Sigma \lambda_j y_j \bmod D$ ou encore

$$\mathcal{P}_p^k(y_i) = \Sigma \lambda_j(\tau x_j) = \tau(\Sigma \lambda_j x_j) \bmod D$$

d'où finalement $\mathcal{P}_p^k(x_i) = \Sigma \lambda_j x_j$, puisque τ est biunivoque. Cela détermine les puissances réduites des éléments universellement transgressifs de $H^*(G, Z_p)$; comme cette dernière est identique à l'algèbre extérieure engendrée par les x_i, les $\mathcal{P}_p^k$ s'y obtiennent alors grâce à 7. 7.

Remarque. On voit que la partie *essentielle* de cette méthode est le calcul des $\mathcal{P}_p^k$ dans $H^*(B_G, Z_p)$; pour en déduire $\mathcal{P}_p^k$ dans $H^*(G, Z_p)$, il nous suffit même de connaître le terme dominant de $\mathcal{P}_p^k(y_i)$, $(1 \leq i \leq l)$. Inversement la connaissance de $\mathcal{P}_p^k$ dans $H^*(G, Z_p)$ détermine le terme dominant de $\mathcal{P}_p^k(y_i)$ mais ne fournit aucun renseignement sur sa partie décomposable. En fait, nous n'aurons besoin que des termes dominants pour toutes les applications données dans la quatrième partie.

11. Le groupe unitaire $\boldsymbol{U}(n)$. Nous expliciterons tout d'abord la méthode générale dans le cas particulier le plus important, celui du groupe unitaire $\boldsymbol{U}(n)$. Nous reprenons les notations du No. 4 ; l'algèbre $H^*(B_{\boldsymbol{U}(n)}, Z_p)$ est engendrée par les classes C_{2i}, $(1 \leq i \leq n)$, et il s'agit essentiellement

d'exprimer $\mathcal{P}_p^k(C_{2i})$ comme polynôme en les C_{2i}; la classe C_{2i} s'identifie à la i-ième fonction symétrique élémentaire $\Sigma x_1 \cdot \cdot \cdot x_i = \sigma_i$, une fois $H^*(B_{U(n)}, Z_p)$ plongé dans $H^*(B_T, Z_p)$ comme il a été dit au No. 3.

LEMME 11.1.

$$\mathcal{P}_p^k(x_1 \cdot \cdot \cdot x_i) = \Sigma_{1 \leq i_1 < \cdots < i_k \leq i} x^{p}_{i_1} \cdot x^{p}_{i_2} \cdot \cdot \cdot \cdot \cdot x^{p}_{i_k} \cdot x_{j_1} \cdot \cdot \cdot \cdot \cdot x_{j_{i-k}}$$

où $\{j_1 < j_2 < \cdot \cdot \cdot < j_{i-k}\}$ *est l'ensemble complémentaire de* $\{i_1, \cdot \cdot \cdot, i_k\}$ *dans la suite* $\{1, 2, \cdot \cdot \cdot, i\}$.

Démonstration par récurrence sur i; pour $i = 1$, le lemme se réduit à la Proposition 8.1; d'après 7.7, (qui vaut ici même si $p = 2$, car $H^*(B_T, Z_p)$ est nulle en toute dimension impaire), on a:

$$\mathcal{P}_p^k(x_1 \cdot \cdot \cdot x_i) = \mathcal{P}_p^k(x_1 \cdot \cdot \cdot x_{i-1}) \cdot x_i + \mathcal{P}_p^{k-1}(x_1 \cdot \cdot \cdot x_{i-1}) \cdot \mathcal{P}_p^1(x_i)$$

et, puisque $P_p^1(x_i) = x_i^p$ d'après 7.5, cela donne:

$$\mathcal{P}_p^k(x_1 \cdot \cdot \cdot x_i) = \mathcal{P}_p^k(x_1 \cdot \cdot \cdot x_{i-1}) \cdot x_i + \mathcal{P}_p^{k-1}(x_1 \cdot \cdot \cdot x_{i-1}) \cdot x_i^p.$$

Vu l'hypothèse de récurrence, le premier terme du second membre est identique à la somme partielle de

$$\Sigma_{1 \leq i_1 < \cdots < i_k \leq i} x^{p}_{i_1} \cdot \cdot \cdot \cdot \cdot x_{i_k}{}^{p} \cdot x_{j_1} \cdot \cdot \cdot \cdot \cdot x_{j_{i-k}}$$

correspondant à $i_k < i$, et le second terme est identique à la somme partielle correspondant à $i_k = i$, ce qui démontre le lemme. (On observera que, si l'on y fait $x_1 = x_2 = \cdot \cdot \cdot = x_i = x$, on retrouve la Proposition 8.1). Il résulte évidemment du lemme 11.1 que

$$\mathcal{P}_p^k(\Sigma\, x_1 \cdot \cdot \cdot x_i) = \Sigma\, x_1{}^{p} \cdot \cdot \cdot x_k{}^{p} x_{k+1} \cdot \cdot \cdot x_i, \tag{11.2}$$

d'où finalement

THÉORÈME 11.3. *Soit* $B_p^{k,j}(\sigma_1, \cdot \cdot \cdot, \sigma_j)$, $(j = i + k(p-1))$, *le polynôme qui exprime le polynôme symétrique de terme typique* $x_1{}^{p} \cdot \cdot \cdot x_k{}^{p} x_{k+1} \cdot \cdot \cdot x_i$ *en fonction des* $\sigma_i = \Sigma\, x_1 \cdot \cdot \cdot x_i$. *Si* $C_{2i} \in H^{2i}(B_{U(n)}, Z_p)$ *désigne la classe de Chern de dimension* $2i$ *réduite mod* p, *on a*

$$\mathcal{P}_p^k(C_{2i}) = B_p^{k,j}(C_2, \cdot \cdot \cdot, C_{2j}).$$

Ce théorème est dû à Wu Wen Tsün [20]; notre démonstration est du reste tout à fait semblable à la sienne, la seule différence étant que chez Wu, l'égalité $C_{2i} = \Sigma\, x_1 \cdot \cdot \cdot x_i$ n'a qu'un caractère "symbolique" et ne peut être utilisée directement pour le calcul de $\mathcal{P}_p^k(C_{2i})$; (Wu raisonne par récurrence

sur n, en utilisant le théorème de dualité rappelé au No. 4, alors qu'ici nous avons interprété les x_i comme des éléments de $H^2(B_T, Z_p)$).

En combinant 10.1 et 11.3, on obtient:

COROLLAIRE 11.4. *Soient $b_p{}^{k,j}\sigma_j$ le terme dominant de $B_p{}^k(\sigma_1, \cdots, \sigma_j)$, et $h_i \in H^{2i-1}(U(n), Z_p)$ l'élément dont l'image par transgression est C_{2i} mod D. On a:*

$$\mathcal{P}_p{}^k(h_i) = b_p{}^{k,j}h_j \qquad (1 \leq i \leq n\,;\, j = i + k(p-1)).$$

(Le terme dominant est bien entendu défini par la condition que

$$B_p{}^{k,j}(\sigma_1, \cdots, \sigma_j) - b_p{}^{k,j}\sigma_j$$

soit un polynôme en $\sigma_1, \cdots, \sigma_{j-1}$.)

12. Cas particuliers. Lorsque k, j, p sont des entiers donnés, le polynôme $B_p{}^{k,j}(\sigma_1, \cdots, \sigma_j)$ et son terme dominant $b_p{}^{k,j}\sigma_j$ peuvent être calculés par un procédé mécanique bien connu; pour $p = 2$, Wu Wen Tsün a même donné une formule générale, valable pour k et j quelconques:

$$B_2{}^{k,j} = \binom{j-k-1}{k}\sigma_j + \binom{j-k-2}{k-1}\sigma_1 \cdot \sigma_{j-1} + \cdots + \binom{j-2k}{1}\sigma_{k-1} \cdot \sigma_{j-k-1} + \sigma_k \cdot \sigma_{j-k};$$

nous ignorons s'il existe une formule générale du même genre pour $p \neq 2$.

Exemples.

1) Calcul de $B_3{}^{1,j}$. Il nous faut calculer $\Sigma\, x_1{}^3x_2 \cdots x_{j-2}$; on a

$$\Sigma\, x_1{}^3x_2 \cdots x_{j-2} = (\Sigma\, x_1{}^2) \cdot (\Sigma\, x_1 \cdots x_{j-2}) - \Sigma\, x_1{}^2x_2 \cdots x_{j-1},$$

$$\Sigma\, x_1{}^2x_2 \cdots x_{j-1} = (\Sigma\, x_1) \cdot (\Sigma\, x_1 \cdots x_{j-1}) - j \cdot \Sigma\, x_1 \cdots x_j.$$

Comme $\Sigma\, x_1{}^2 = (\sigma_1)^2 - 2\sigma_2$, on trouve en définitive:

12.1 $$B_3{}^{1,j} = (\sigma_1)^2 \cdot \sigma_{j-2} - 2\sigma_2 \cdot \sigma_{j-2} - \sigma_1 \cdot \sigma_{j-1} + j \cdot \sigma_j,$$

d'où, compte tenu de 11.3:

12.2 $$\mathcal{P}_3{}^1(C_{2j-4}) = (C_2)^2 \cdot C_{2j-4} - 2 \cdot C_4 \cdot C_{2j-4} - C_2 \cdot C_{2j-2} + j \cdot C_{2j}.$$

2) Calcul de $b_p{}^{1,j}$. L'exemple précédent montre que $b_3{}^{1,j} = j$; nous allons voir que plus généralement:

12.3 $$b_p{}^{1,j} \equiv j \mod p.$$

Puisque $(-1)^{p+1} \equiv 1 \bmod p$ pour tout p premier, il suffit évidemment

de prouver que le terme dominant de $\Sigma x_1^q x_2 \cdot \cdot \cdot x_{j-q+1}$ est $(-1)^{q+1} j \cdot \sigma_j$, ce qui, pour $q = 2$, résulte de la formule:

$$\Sigma x_1^2 x_2 \cdot \cdot \cdot x_{j-1} = (\Sigma x_1)(\Sigma x_1 \cdot \cdot \cdot x_{j-1}) - j \cdot \Sigma x_1 \cdot \cdot \cdot x_j,$$

et, pour $q > 2$, se démontre par récurrence sur q à l'aide de l'identité:

$$\Sigma x_1^q x_2 \cdot \cdot \cdot x_{j-q+1} = (\Sigma x_1^{q-1})(\Sigma x_1 \cdot \cdot \cdot x_{j-q+1}) - \Sigma x_1^{q-1} x_2 \cdot \cdot \cdot x_{j-q+2}.$$

En combinant 12.3 avec 11.3 et 11.4, dont nous gardons les notations, on obtient:

PROPOSITION 12.4. *Soient p un nombre premier, j un entier $> p$, non divisible par p. La classe de Chern C_{2j}, réduite mod p, est égale à $1/j \cdot \mathcal{P}_p^1(C_{2j-2p+2})$ augmentée d'un polynôme par rapport aux classes C_{2i}, $(i < j)$, et l'on a $h_j = 1/j \cdot \mathcal{P}_p^1(h_{j-p+1})$.*

En fait, dans la plupart des applications, c'est j qui est donné, et l'on cherche un nombre premier p vérifiant les hypothèses de 12.4; on a à ce sujet:

LEMME 12.5. *Pour tout entier $j \geq 3$, il existe un nombre premier p tel que $p < j$ et que $j \not\equiv 0 \bmod p$; ce nombre peut être choisi impair si $j \geq 4$.*

La première partie se démontre en prenant pour p un diviseur premier de $j - 1$, la seconde en prenant pour p un diviseur premier de $j - 1$ ou de $j - 2$ suivant que j est pair ou impair.

Il résulte de 12.4 et 12.5:

PROPOSITION 12.6. *Soit j un entier ≥ 3. Il existe un nombre premier $p < j$ ne divisant pas j, impair si $j \geq 4$, tel que C_{2j}, (resp. h_j), réduite mod p, soit égale à la somme d'un polynôme par rapport aux classes C_{2i}, (resp. h_i), $i < j$, et de $\mathcal{P}_p^1(\lambda C_{2j-2p+2})$, (resp. $\mathcal{P}_p^1(\lambda h_{j-p+1})$), $\lambda \in Z_p$.*

Puisque tout élément de $H^{2j}(B_{\boldsymbol{U}(n)}, Z_p)$, (resp. de $H^{2j-1}(\boldsymbol{U}(n), Z_p)$, est somme d'un multiple de C_{2j}, (resp. de h_j), et d'un polynôme en les C_{2i}, (resp. en les h_i), $i < j$, on déduit de 12.6:

COROLLAIRE 12.7. *Si j et p vérifient les conditions de 12.6, tout élément de $H^{2j}(B_{\boldsymbol{U}(n)}, Z_p)$, et tout élément de $H^{2j-1}(\boldsymbol{U}(n), Z_p)$, peuvent s'exprimer à l'aide de cup-produits et d'opérations de Steenrod à partir d'éléments de dimensions strictement plus petites.*

On notera que ce Corollaire vaut aussi bien pour $\boldsymbol{SU}(n)$; cela pourrait se montrer par des calculs analogues, mais il est plus simple de remarquer que, $\boldsymbol{SU}(n)$ étant totalement nom homologue à zéro dans $\boldsymbol{U}(n)$, les algèbres

$H^*(\boldsymbol{SU}(n), Z_p)$ et $H^*(B_{\boldsymbol{SU}(n)}, Z_p)$ sont des quotients de $H^*(\boldsymbol{U}(n), Z_p)$ et $H^*(B_{\boldsymbol{U}(n)}, Z_p)$, ([2], Cor. à la Prop. 21. 3).

13. Le groupe symplectique unitaire $Sp(n)$. Si l'on applique la méthode générale, on est amené à calculer $\mathcal{P}_p^k(K_{4i})$ avec $K_{4i} = \Sigma\, x_1^2 \cdot \cdots \cdot x_i^2$.

Le lemme 11. 1 donne la valeur de $\mathcal{P}_p^k(x_1^2 \cdot \cdots \cdot x_i^2)$, d'où celle de $\mathcal{P}_p^k(K_{4i})$; tous calculs faits, on trouve :

13. 1 $$\mathcal{P}_p^k(\Sigma\, x_1^2 \cdot \cdots \cdot x_i^2) = \Sigma_{2r+s=k}\, 2^s \Sigma\, x_1^{2p} \cdot x_2^{2p} \cdot \cdots \cdot x_r^{2p} \cdot x_{r+1}^{p+1} \cdot \cdots \cdot x_{r+s}^{p+1} \cdot x_{r+s+1}^2 \cdot \cdots \cdot x_i^2.$$

Lorsque i et k sont assez petits, cette formule permet d'exprimer $\mathcal{P}_p^k(K_{4i})$ comme polynôme en les K_{4i}.

Exemples.

1) $k = 1$, $p = 3$. On doit calculer $2 \cdot \Sigma\, x_1^4 x_2^2 \cdot \cdots \cdot x_i^2$. On a

$$\Sigma\, x_1^4 x_2^2 \cdot \cdots \cdot x_i^2 = (\Sigma\, x_1^2)(\Sigma\, x_1^2 \cdot \cdots \cdot x_i^2) - (i+1)\Sigma\, x_1^2 \cdot \cdots \cdot x_{i+1}^2,$$

ce qui donne :

13. 2 $$\mathcal{P}_3^1(K_{4i}) = 2 \cdot K_4 \cdot K_{4i} - (2i+2) \cdot K_{4i+4}.$$

2) $k = 1$, $p = 5$. On doit calculer $2\,\Sigma\, x_1^6 x_2^2 \cdot \cdots \cdot x_i^2$. On a

$$\Sigma\, x_1^6 x_2^2 \cdot \cdots \cdot x_i^2 = (\Sigma\, x_1^4)(\Sigma\, x_1^2 \cdot \cdots \cdot x_i^2) - \Sigma\, x_1^4 x_2^2 \cdot \cdots \cdot x_{i+1}^2,$$

ce qui, compte tenu du calcul précédent et de $(\Sigma\, x_1^4) = (\Sigma\, x_1^2)^2 - 2 \cdot \Sigma x_1^2 x_2^2$ donne :

13. 3 $$\mathcal{P}_5^1(K_{4i}) = 2 \cdot K_4^2 \cdot K_{4i} + K_8 \cdot K_{4i} + 3 \cdot K_4 \cdot K_{4i+4} + (2i+4) \cdot K_{4i+8}.$$

En fait, il est en général plus commode d'utiliser le plongement canonique de $\boldsymbol{Sp}(n)$ dans $\boldsymbol{U}(2n)$; il conduit à un homomorphisme

$$\nu : H^*(B_{\boldsymbol{U}(2n)}) \to H^*(B_{\boldsymbol{Sp}(n)})$$

qui, d'après 5. 4, applique C_{4i+2} sur zéro et C_{4i} sur $(-1)^i K_{4i}$. On a donc :

$$\mathcal{P}_p^k(K_{4i}) = (-1)^i \mathcal{P}_p^k(\nu(C_{4i})) = (-1)^i \nu(\mathcal{P}_p^k(C_{4i})).$$

Appliquant alors le Théorème 11. 3, on trouve :

THÉORÈME 13. 4. *Avec les notations du Théorème* 11. 3, *on a :*

$$\mathcal{P}_p^k(K_{4i}) = (-1)^i B_p^{k,2j}(0, -K_4, 0, K_8, \cdots, 0, (-1)^t K_{4t}, \cdots, (-1)^j K_{4j}),$$

j désignant l'entier $i + k(p-1)/2$.

(On a supposé $p \neq 2$, ou bien k pair, car sinon il est évident que $\mathcal{P}_p^k(K_{4i}) = 0$, puisque $H^{4i+2k}(B_{Sp(n)}, Z_2) = 0$.)

De ce théorème on tire la conséquence suivante, analogue à 11.4:

COROLLAIRE 13.5. *Si v_i désigne l'élément de $H^{4i-1}(\mathbf{Sp}(n), Z_p)$ dont l'image par transgression est K_{4i} mod D, on a:*

$\mathcal{P}_p^k(v_i) = 0$ *si* $p = 2$ *et si* k *est impair,*

$\mathcal{P}_p^k(v_i) = (-1)^{k(p-1)/2} \cdot b_p^{k,2j} \cdot v_j$ *sinon, (avec* $j = i + k(p-1)/2$*).*

(On pourrait également déduire 13.5 de 11.4 et du fait que v_i est induit par $(-1)^i h_{2i}$, cf. No. 5).

Enfin, 13.4 et 13.5, joints à l'égalité $b_p^{1,2j} \equiv 2j \bmod p$, donnent l'analogue suivant de 12.7:

COROLLAIRE 13.6. *Soit j un entier > 2. Il existe un nombre premier $p < 2j$ impair tel que tout élément de $H^{4j}(B_{Sp(n)}, Z_p)$ et tout élément de $H^{4j-1}(\mathbf{Sp}(n), Z_p)$ puissent s'exprimer à l'aide de cup-produits et d'opérations de Steenrod à partir d'éléments de dimensions strictement plus petites.*

14. Le groupe orthogonal $SO(n)$. Nous devons nous borner ici aux nombres premiers p *impairs*, puisque $\mathbf{SO}(n)$ a de la 2-torsion. On a vu au No. 5 que $H^*(\mathbf{SO}(n), Z_p)$ admet comme système de générateurs les classes de Pontrjagin réduites P_{4i}, auxquelles s'ajoute la classe de Stiefel-Whitney W_n lorsque n est pair; de plus le plongement canonique de $\mathbf{SO}(n)$ dans $\mathbf{U}(n)$ définit un homomorphisme $\sigma: H^*(B_{U(n)}, Z_p) \to H^*(B_{SO(n)}, Z_p)$ qui applique C_{4i+2} sur 0 et C_{4i} sur $(-1)^i P_{4i}$, (*voir* 5.3). On peut alors répéter au sujet des P_{4i} le raisonnement fait au No. 13 pour les K_{4i}, et l'on obtient:

THÉORÈME 14.1. *Si $P_{4i} \in H^{4i}(B_{SO(n)}, Z_p)$ désigne la classe de Pontrjagin de dimension $4i$ réduite mod p, $(p \neq 2)$, on a:*

$$\mathcal{P}_p^k(P_{4i}) = (-1)^i \cdot B_p^{k,2j}(0, -P_4, 0, P_8, \cdots, (-1)^j P_{4j}),$$
$$(j = i + k(p-1)/2).$$

Dans le cas où $n = 2m$ est pair, il reste encore à calculer $\mathcal{P}_p^k(W_{2m})$; le plus simple est ici d'appliquer la méthode générale; W_{2m} étant égale à $x_1 \cdots x_m$, on a $\mathcal{P}_p^k(W_{2m}) = \Sigma\, x_1^p \cdots x_k^p x_{k+1} \cdots x_m$, ou encore puisqu'il n'y a que m lettres $x_1 \cdots x_m$:

$$\mathcal{P}_p^k(W_{2m}) = x_1 \cdots x_m \Sigma\, x_1^{p-1} \cdots x_k^{p-1} = W_{2m} \Sigma\, x_1^{2h} \cdots x_k^{2h},$$

en posant $h = (p-1)/2$; il nous reste donc à exprimer $\Sigma\, x_1^{2h} \cdots x_k^{2h}$ comme

polynôme en les $P_{4i} = \Sigma_1{}^2 \cdots x_i{}^2$ et $(W_{2m})^2$, ce qui conduit au théorème suivant:

THÉORÈME 13. 2. *Soient p un nombre premier impair, $h = (p-1)/2$, et $C^{k,h}(\sigma_1, \cdots, \sigma_m)$ le polynôme qui exprime la fonction symétrique $\Sigma\, x_1{}^h \cdots x_k{}^h$ comme polynôme en les $\sigma_i = \Sigma\, x_1 \cdots x_i$, les lettres x_i étant au nombre de m. Si $W_{2m} \in H^{2m}(B_{SO(2m)}, Z_p)$ désigne la classe de Stiefel-Whitney de dimension $2m$, réduite mod p, on a:*

$$\mathcal{P}_p{}^k(W_{2m}) = W_{2m} \cdot C^{k,h}(P_4, P_8, \cdots, P_{4m-4}, (W_{2m})^2).$$

En particulier, on observera que $\mathcal{P}_p{}^k(W_{2m})$ est toujours un élément *décomposable* si $k \geq 1$.

Les théorèmes 13. 1 et 13. 2 entraînent:

COROLLAIRE 13. 3. *Soient p un nombre premier impair, t_i l'élément de $H^{4i-1}(\boldsymbol{SO}(n), Z_p)$ dont l'image par transgression est P_{4i} mod D, et, pour n pair, $u_n \in H^{n-1}(\boldsymbol{SO}(n), Z_p)$ celui dont l'image par transgression est W_n mod D. On a:*

$$\mathcal{P}_p{}^k(u_n) = 0 \text{ lorsque } k \geq 1.$$

$$\mathcal{P}_p{}^k(t_i) = (-1)^{k(p-1)/2} \cdot b_p{}^{k,2j} \cdot t_j, \qquad (j = i + k(p-1)/2).$$

(Remarquons en passant que l'on aurait pu prévoir *a priori* la nullité de $\mathcal{P}_p{}^k(u_n)$, $(k \geq 1)$. En effet, par définition même, W_n est image par transgression de l'élément de $H^{n-1}(\boldsymbol{SO}(n), Z)$ qui est image de la classe fondamentale de $\boldsymbol{S}_{n-1}$ par l'homomorphisme transposé de la projection naturelle de $\boldsymbol{SO}(n)$ sur $\boldsymbol{S}_{n-1}$; cet élément, une fois réduit mod p, est forcément égal à u_n puisque la transgression est ici biunivoque; $\mathcal{P}_p{}^k(u_n)$ est donc l'image d'un élément de $H^{n-1+2k(p-1)}(\boldsymbol{S}_{n-1}, Z_p)$ et est bien nul si $k \geq 1$.)

Enfin, par une démonstration tout à fait semblable à celle de 12. 7, on déduit de ce qui précède:

PROPOSITION 13. 4. *Soient j un entier ≥ 2, n un entier impair. Il existe un nombre premier impair $p < 2j$ tel que tout élément de $H^{4j}(B_{SO(n)}, Z_p)$ et tout élément de $H^{4j-1}(\boldsymbol{SO}(n), Z_p)$ puissent s'exprimer à l'aide de cup-produits et d'opérations de Steenrod à partir d'éléments de dimensions strictement plus petites.*

Remarque. Soit G un groupe classique, et soit $2q-1$ la plus grande dimension pour laquelle $H^*(G, R)$, (R corps des nombres réels), contient un élément universellement transgressif non nul.

Quel que soit le nombre premier impair $p < q$, l'opération $\mathcal{P}_p^1$ est non triviale dans $H^*(G, Z_p)$, et de façon plus précise, $\mathcal{P}_p^1$ transforme un élément $x \in H^3(G, Z_p)$ universellement transgressif et non nul, en un élément non nul de $H^{2p+1}(G, Z_p)$.

Cela se vérifie sur chaque cas particulier $G = \boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$, $\boldsymbol{SO}(n)$, en tenant compte de $b_p^{1,j} \equiv j \bmod p$; pour $G = \boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$ c'est encore vrai si $p = 2$.

Par contre, si $p > q$, il est évident a priori que $\mathcal{P}_p^1$, et, plus généralement, $\mathcal{P}_p^k$, est nul dans $H^*(G, Z_p)$.

IV. Applications.

15. Les sphères presque complexes et les algèbres à division sur le corps des nombres réels.

PROPOSITION 15.1. *La sphère $\boldsymbol{S}_{2n}$, $(n \geq 4)$, n'admet pas de structure presque complexe.*

Raisonnons par l'absurde et soient $c_{2i} \in H^{2i}(\boldsymbol{S}_{2n}, Z)$ les classes de Chern d'une structure presque complexe de $\boldsymbol{S}_{2n}$; la classe c_{2i} est l'image de la classe $C_{2i} \in H^{2i}(B_{\boldsymbol{U}(n)}, Z)$ par l'homomorphisme transposé d'une certaine application continue de $\boldsymbol{S}_{2n}$ dans $B_{\boldsymbol{U}(n)}$, $(1 \leq i \leq n)$; il existe donc, d'après 12.6 où l'on fait $j = n$, un nombre premier *impair* $p < n$ tel que c_{2n}, réduite mod p, s'exprime à l'aide de cup produits et de l'opération de Steenrod $\mathcal{P}_p^1$ à partir des c_{2i}, $(i < n)$. Mais ces dernières sont nulles puisque $H^{2i}(\boldsymbol{S}_{2n}, Z) = 0$ pour $0 < i < n$, ce qui montre que $c_{2n} \equiv 0 \bmod p$.

Soit d'autre part h la classe fondamentale de $H^{2n}(\boldsymbol{S}_{2n}, Z)$; on sait ([14], 41.8), que c_{2n} est égale à $\chi(\boldsymbol{S}_{2n}) \cdot h$, c'est à dire à $2 \cdot h$. Comme on a choisi p impair, on voit que $c_{2n} \not\equiv 0 \bmod p$, d'où une contradiction.

Remarque. Le raisonnement précédent s'applique à la sphère $\boldsymbol{S}_{2n}$ munie *d'une structure différentiable quelconque*, alors que la démonstration classique d'inexistence d'une structure presque-complexe sur $\boldsymbol{S}_4$ ([14], 41.20), suppose de façon essentielle que $\boldsymbol{S}_4$ est munie de la structure usuelle.

On sait que $\boldsymbol{S}_6$ admet une structure presque complexe définie à l'aide des octaves de Cayley ([14], 41.21); en généralisant cette construction, nous allons démontrer la proposition suivante:

PROPOSITION 15.2. *Soit A une algèbre (non necessairement associative) sur le corps des nombres réels R, jouissant des propriétés suivantes:*

(a) *A possède un élémente unité, qui sera noté e.*

(b) *La relation $a \cdot b = 0$ entraîne $a = 0$ ou $b = 0$.*

(c) *La relation $a \cdot b = e$ entraîne que a, b et e vérifient une relation linéaire à coefficients réels.*

Dans ces conditions, la dimension de l'algèbre A est 1, 2, 4, *ou* 8.

Avant de passer à la démonstration remarquons que, d'après Hopf et Stiefel,[5] les conditions (a) et (b) seules permettent d'établir que la dimension de A est une puissance de deux; on ignore si elles impliquent l'inégalité $\dim A \leq 8$. Dans le cas où la sous-algèbre engendrée par un élément quelconque est associative la condition (c) équivaut à dire que tout élément de A vérifie une relation quadratique.

Posons $n = \dim A$, et supposons $n \geq 3$; nous devons montrer que $n = 4$ ou $n = 8$. Introduisons sur A, considéré comme espace vectoriel réel, une forme quadratique définitive positive. Soient H l'hyperplan homogène de A orthogonal à e, (relativement au produit scalaire défini par cette forme), $\boldsymbol{S}$ l'ensemble des points de H à distance unité de l'origine; $\boldsymbol{S}$ est donc une sphère de dimension $n - 2$. Si x est un point de $\boldsymbol{S}$ on peut identifier l'espace vectoriel T_x des vecteurs tangents à $\boldsymbol{S}$ en x au sous-espace de A orthogonal à e et x et de dimension $n - 2$; soit k_x l'opération de projection orthogonale de A sur T_x, et posons

$$J_x(y) = k_x(x \cdot y) \text{ pour } y \in T_x.$$

L'opérateur J_x est un endomorphisme de T_x qui varie continûment avec x; montrons que J_x *n'a pas de valeur propre réelle*: une égalité $J_x(y) = \lambda y$, $(\lambda \in R)$, peut s'écrire $k_x(x \cdot y - \lambda e) = 0$, ou encore $x \cdot y - \lambda y = \mu e + \nu x$, $(\mu, \nu \in R)$, ce qui donne $(x - \lambda \cdot e)(y - \nu \cdot e) = (\mu + \lambda \cdot \nu)e$. Si l'on suppose $y \neq 0$, les éléments $x - \lambda \cdot e$ et $y - \nu \cdot e$ sont $\neq 0$ puisque e et y sont orthogonaux à e; il s'ensuit d'après (b) que $(\mu + \lambda \cdot \nu) \neq 0$; on peut donc poser $w = (\mu + \lambda \cdot \nu)^{-1} \cdot (y - \nu e)$ et l'on a $(x - \lambda \cdot e) \cdot w = e$; d'après (c), il existe alors une relation linéaire entre $(x - \lambda \cdot e)$, w et e, donc aussi entre x, y et e; mais cela est absurde puisque ces éléments sont deux à deux orthogonaux.

Ainsi, nous avons associé à tout point $x \in \boldsymbol{S}$ un endomorphisme sans valeurs propres réelles J_x de l'espace des vecteurs tangents à $\boldsymbol{S}$ en x; il en résulte classiquement (voir ci-dessous) que $\boldsymbol{S}$ peut être munie d'une structure presque complexe, et l'on a donc $n - 2 = 2$ ou 6, c'est à dire $n = 4$ ou 8.

Note. Indiquons encore, pour être complet, comment on passe de l'exis-

[5] **E. Stiefel**, *Comm. Math. Helv.*, **tome 13 (1940-41), pp. 201-218, H. Hopf,** *ibid.*, **pp. 219-239.**

tence de l'endomorphisme J_x à une structure presque complexe sur $\mathbf{S}$.[6] Il nous faut remplacer l'endomorphisme J_x par un endomorphisme I_x tel que $(I_x)^2 = -1$.

Soit $T_x \otimes C$ l'extension complexe de l'espace vectoriel réel T_x et soient $(\alpha_1, \cdots, \alpha_q, \bar{\alpha}_1, \cdots, \bar{\alpha}_q)$ les valeurs propres de J_x dans $T_x \otimes C$, chaque α_i ayant une partie imaginaire positive; pour toute valeur propre α nous désignons par V_α le plus grand sous-espace vectoriel de $T_x \otimes C$ sur lequel $J_x - \alpha$ est nilpotent; on sait que $T_x \otimes C$ est la somme directe des V_α, et il est clair que $V_{\bar{\alpha}} = \bar{V}_\alpha$; l'espace $T_x \otimes C$ est donc la somme directe de $W = V_{\alpha_1} + \cdots + V_{\alpha_q}$ et de $\bar{W}$, et tout élément $y \in T_x$ peut se mettre d'une seule façon sous la forme $y = w + \bar{w}$, $(w \in W)$. Posons alors $I_x(y) = iw - i\bar{w}$, c'est un élément de T_x et ainsi I_x définit un endomorphisme de T_x qui vérifie visiblement la condition $(I_x)^2 = -1$; il reste encore à s'assurer qu'il varie continûment avec x; cela résulte, par exemple, de la continuité des valeurs propres de J_x.

16. Sur les espaces fibrés à base sphérique. Nous intercalons ici quelques résultats dont nous aurons besoin dans les Nos. suivants; la Proposition 16.1 a été également utilisée par Miller [10].

PROPOSITION 16.1. *Soient E un espace fibré de fibre F, de base $\mathbf{S}_r$, β la classe fondamentale de $H^r(\mathbf{S}_r, Z_p)$, (p premier), ζ la projection de E sur $\mathbf{S}_r$. Si la classe $\gamma = \zeta^*(\beta)$ peut s'exprimer à l'aide de cup-produits et d'opérations de Steenrod à partir d'éléments de $H^*(E, Z_p)$ de dimensions $< r$, l'espace fibré E n'admet pas de section.*

Par hypothèse on peut trouver des éléments $u_1, \cdots, u_k \in H^*(E, Z_p)$, $(0 < \dim u_i < r)$, tels que $\gamma = f(u_1, \cdots, u_k)$, où f est une expression formée à l'aide de cup-produits et d'operations $\mathcal{P}_p^k$. Si $s : \mathbf{S}_r \to E$ était une section de l'espace fibré E, l'homomorphisme $s^* : H^*(E, Z_p) \to H^*(\mathbf{S}_r, Z_p)$ vérifierait la relation $s^* \circ \zeta^* = 1$, et l'on aurait:

$$\beta = s^* \circ \zeta^*(\beta) = s^*(\gamma) = s^*(f(u_1, \cdots, u_k)) = f(s^*(u_1), \cdots, s^*(u_k)) = 0$$

puisque $s^*(u_i)$ est une classe de cohomologie de $\mathbf{S}_r$ de dimension strictement comprise entre 0 et r; mais comme $\beta \neq 0$, cela est impossible et montre qu'il n'y a pas de section.

(Bien entendu, ce genre de raisonnement a une portée plus générale et s'applique à d'autres espaces que les sphères, pourvu que l'on soit certain que $s^*(u_i) = 0$ pour tout i.)

La Proposition 16.1 revient à dire que *la classe caractéristique* $\alpha \in \pi_{r-1}(F)$

[6] La démonstration qui suit nous a été obligeamment communiquée par G. de Rham.

de la fibration considérée est un élément non nul de $\pi_{r-1}(F)$; nous allons préciser ce résultat dans les deux propositions suivantes.

PROPOSITION 16.2. *Avec les hypothèses et notations précédentes, la classe caractéristique* $\alpha \in \pi_{r-1}(F)$ *est, soit d'ordre infini, soit d'ordre fini divisible par* p.

Il nous faut montrer que l'on a $q \cdot \alpha \neq 0$ dans $\pi_{r-1}(F)$ lorsque q est un entier $\neq 0$ et non divisible par p.

Pour cela, soient $\psi : \boldsymbol{S}_r \to \boldsymbol{S}_r$ une application de degré q, E' l'espace fibré image réciproque de E par ψ, et ζ' la projection de E' sur $\boldsymbol{S}_r$; il existe donc une application $\bar{\psi} : E' \to E$ telle que $\psi \circ \zeta' = \zeta \circ \bar{\psi}$. On a évidemment $\psi^*(\beta) = q \cdot \beta$, d'où

$$q \cdot \zeta'^*(\beta) = \zeta'^* \circ \psi^*(\beta) = \bar{\psi}^* \circ \zeta^*(\beta) = \bar{\psi}^*(f(u_1, \cdots, u_k)),$$

ce que donne, puisque $q \not\equiv 0 \bmod p$,

$$\zeta'^*(\beta) = 1/q \cdot f(\bar{\psi}^*(u_1), \cdots, \bar{\psi}^*(u_k)),$$

et E' n'a pas de section d'après 16.1; cela signifie que la classe caractéristique $\alpha' \in \pi_{r-1}(F)$ de E' est non nulle, et comme cette classe est évidemment égale à $q \cdot \alpha$, la proposition est démontrée.

PROPOSITION 16.3. *Ajoutons aux hypothèses de* 16.1 *les suivantes:*

(a) *L'espace fibré* E *est un espace fibré principal à groupe structural* F *connexe par arcs.*

(b) *Si* $u_1, \cdots, u_k$ *sont les éléments de* $H^*(E, Z_p)$ *tels que* $\gamma = f(u_1, \cdots, u_k)$, *on a* $0 < \dim u_i \leq r - 2$ *pour tout* i.

(c) *La classe* γ *est* $\neq 0$.

Alors il n'existe aucun élément $\alpha' \in \pi_{r-1}(F)$ *tel que* $\alpha = p \cdot \alpha'$.

(Autrement dit, α définit un élément non nul de $\pi_{r-1}(F) \otimes Z_p$, résultat évidemment plus précis que celui de 16.2.)

Nous raisonnons par l'absurde et supposons donc l'existence d'un élément $\alpha' \in \pi_{r-1}(F)$ tel que $\alpha = p \cdot \alpha'$. On sait ([14], 18.5), que si l'on associe à tout espace fibré principal de base $\boldsymbol{S}_r$, de fibre F, sa classe caractéristique, on définit une correspondance biunivoque entre les classes d'espaces fibrés principaux de fibre F, base $\boldsymbol{S}_r$, et les éléments de $\pi_{r-1}(F)$. Vu notre hypothèse, il existe donc un espace fibré principal E', de classe caractéristique α', dont E est l'image réciproque par une application $\sigma : \boldsymbol{S}_r \to \boldsymbol{S}_r$ de degré p. On désigne comme dans la démonstration précédente par ζ' la projection de E' sur $\boldsymbol{S}_r$; soient encore $\bar{\sigma} : E \to E'$ l'application canonique de E dans E'

et $i^*: H^*(E, Z_p) \to H^*(F, Z_p)$, resp. $i'^*: H^*(E', Z_p) \to H^*(F, Z_p)$, l'homomorphisme défini par l'injection d'une fibre dans E, resp. E'.

On a évidemment $i^* \circ \bar{\sigma}^* = i'^*$; mais, d'après la suite exacte de H. C. Wang (*Duke Math. Jour.*, vol. 16 (1949), 33-38, ou [12], p. 471), i^* et i'^* sont des isomorphismes sur pour les dimensions $\leq r-2$; il en est donc de même pour $\bar{\sigma}^*$ et vu l'hypothèse (b), $H^*(E', Z_p)$ contient des éléments u'_i tels que $u_i = \bar{\sigma}^*(u'_i)$, $(1 \leq i \leq k)$. On a donc

$$i^*(\gamma) = i^*(f(u_1, \cdots, u_k)) = i^* \circ \bar{\sigma}^*(f(u'_1, \cdots, u'_k)) = i'^*(f(u'_1, \cdots, u'_k)).$$

Il est clair que $i^*(\gamma) = 0$, donc $i'^*(f(u'_1, \cdots, u'_k)) = 0$, et la suite de Wang donne alors $f(u'_1, \cdots, u'_k) = \lambda \cdot \zeta'^*(\beta)$, $(\lambda \in Z_p)$. On en tire

$$\gamma = \lambda \cdot \bar{\sigma}^* \circ \zeta'^*(\beta) = \lambda \cdot \zeta^* \circ \sigma^*(\beta),$$

mais cela est impossible car $\sigma^*(\beta) = 0$ puisque σ est de degré p, et d'autre part $\gamma \neq 0$ d'après (c), ce qui démontre 16.3.

17. Inexistence de sections dans certains espaces fibrés.

PROPOSITION 17.1. *Les fibrations suivantes n'ont pas de section:*

(a) $\boldsymbol{SU}(n)/\boldsymbol{SU}(n-1) = \boldsymbol{S}_{2n-1}$ *pour* $n \geq 3$.

(b) $\boldsymbol{U}(n)/\boldsymbol{U}(n-1) = \boldsymbol{S}_{2n-1}$ *pour* $n \geq 3$.

(c) $\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-1) = \boldsymbol{S}_{4n-1}$ *pour* $n \geq 2$.

(d) $\boldsymbol{Spin}(9)/\boldsymbol{Spin}(7) = \boldsymbol{S}_{15}$.

(e) $\boldsymbol{Spin}(7)/\boldsymbol{G}_2 = \boldsymbol{S}_7$.

(Les fibrations (a), (b), (c) sont classiques, pour les deux dernières *voir* [1].)

D'après 16.1, il suffit de trouver dans chaque cas un nombre premier p tel que tout élément de $H^{2n-1}(\boldsymbol{SU}(n), Z_p)$, de $H^{2n-1}(\boldsymbol{U}(n), Z_p)$, de $H^{4n-1}(\boldsymbol{Sp}(n), Z_p)$, de $H^{15}(\boldsymbol{Spin}(9), Z_p)$, de $H^7(\boldsymbol{Spin}(7), Z_p)$ s'exprime à l'aide de cup-produits et de puissances réduites à partir d'éléments de dimensions strictement inférieures. C'est possible pour $\boldsymbol{U}(n)$ et $\boldsymbol{SU}(n)$ d'après 12.7 et pour $\boldsymbol{Sp}(n)$ d'après 13.6. Dans le cas (d), on choisit d'abord p impair tel que tout élément de $H^{15}(\boldsymbol{SO}(9), Z_p)$ s'exprime à l'aide de cup-produits et d'opérations $\mathcal{P}_p^k$ à partir d'éléments de dimensions < 15, ce qui est possible pour $p = 3, 5, 7$ d'après 13.4; puisque $\boldsymbol{Spin}(9)$ est un revêtement à deux feuillets de $\boldsymbol{SO}(9)$, l'homomorphisme $H^*(\boldsymbol{SO}(9), Z_p) \to H^*(\boldsymbol{Spin}(9), Z_p)$ transposé de la projection est un isomorphisme sur, et la même propriété a lieu dans $H^*(\boldsymbol{Spin}(9), Z_p)$; on raisonne de la même façon dans le cas (e), pour $p = 3$.

Remarque. La Prop. 17.1 (a) montre que $\boldsymbol{SU}(3)/\boldsymbol{SU}(2) = \boldsymbol{S}_5$ n'a pas de section, autrement dit que la classe caractéristique $\alpha \in \pi_4(\boldsymbol{S}_3)$ de cette fibration est non nulle, ce qui a été tout d'abord démontré par Pontrjagin [11], par une étude homotopique de α; on observera que nous sommes parvenus à ce résultat par voie cohomologique, (en utilisant l'opération Sq^2).

PROPOSITION 17.2. (a) *La classe caractéristique de la fibration* $\boldsymbol{SU}(n)/\boldsymbol{SU}(n-1) = \boldsymbol{S}_{2n-1}$ *définit un élément non nul de* $\pi_{2n-2}(SU(n-1)) \otimes Z_p$ *pour tout* p *premier* $< n$, *ne divisant pas* n.

(b) *Il en est de même pour la fibration* $\boldsymbol{U}(n)/\boldsymbol{U}(n-1) = \boldsymbol{S}_{2n-1}$.

(c) *Il en est de même pour tout* p *premier impair* $< 2n$, *ne divisant pas* n, *pour la fibration* $\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-1) = \boldsymbol{S}_{4n-1}$.

(d) *La classe caractéristique de la fibration* $\boldsymbol{Spin}(9)/\boldsymbol{Spin}(7) = \boldsymbol{S}_{15}$ *définit un élément non nul de* $\pi_{14}(Spin(7)) \otimes Z_p$ *pour* $p = 3, 5, 7$.

(e) *La classe caractéristique de la fibration* $\boldsymbol{Spin}(7)/\boldsymbol{G}_2 = \boldsymbol{S}_7$ *définit un élément non nul de* $\pi_6(G_2) \otimes Z_3$.

On doit montrer que les hypothèses de 16.3 sont vérifiées; 16.3(a) est évidente, 16.3(b) résulte de ce que l'opération f utilisée pour établir 17.1 est chaque fois $\mathcal{P}_p^1$, qui augmente le degré $2(p-1) \geq 2$ unités. Enfin, il est évident dans chaque cas envisagé ici que γ est universellement transgressif, et il résulte de la détermination même des algèbres $H^*(E, Z_p)$ que $\gamma \neq 0$; 16.3(c) est donc aussi satisfaite.

On vérifie ensuite, en utilisant la formule $b_p^{1,j} \equiv j \bmod p$ et les résultats de III que dans chaque cas, les nombres premiers de l'énoncé sont tels que $\gamma = \mathcal{P}_p^1(u)$, $(u \in H^*(E, Z_p))$; 17.2 résulte donc de 16.3.

Remarque. Pour démontrer la Proposition 17.2, nous n'avons eu besoin que des opération $\mathcal{P}_p^1$. En se servant des puissances réduites $\mathcal{P}_p^k$ pour k quelconque, et d'un lemme sur les coefficients $b_p^{k,j}$ qui sera établi plus loin (lemme 20.7), on déduit par le raisonnement ci-dessus un résultat plus complet, que nous énoncerons uniquement dans le cas (a) pour simplifier:

La classe caractéristique de $\boldsymbol{SU}(n)/\boldsymbol{SU}(n-1) = \boldsymbol{S}_{2n-1}$ *définit un élément non nul de* $\pi_{2n-2}(\boldsymbol{SU}(n-1)) \otimes Z_p$ *lorsque le nombre premier* $p < n$ *vérifie la condition suivante: Si* $h(p, n)$ *est le plus grand entier tel que* $p^{h(p,n)} \cdot (p-1) < n$, *le nombre* n *n'est pas divisible par* $p^{h(p,n)+1}$.

Nous terminons le No. 17 par la Proposition suivante, tout à fait analogue au cas (c) des Prop. 17.1 et 17.2:

PROPOSITION 17.3. *Soit* $\boldsymbol{W}_{4n-1}$ *la variété des vecteurs de longueur unité tangents à la sphère* $\boldsymbol{S}_{2n}$. *La fibration* $\boldsymbol{SO}(2n+1)/\boldsymbol{SO}(2n-1) = \boldsymbol{W}_{4n-1}$

n'a pas de section si $n \geq 2$. *Si* p *est un nombre premier impair* $< 2n$, *ne divisant pas* n, *l'homomorphisme bord* d *de* $\pi_{4n-1}(\boldsymbol{W}_{4n-1})$ *dans* $\pi_{4n-2}(\boldsymbol{SO}(2n-1))$ *définit un sous-groupe isomorphe à* Z_p *de* $\pi_{4n-2}(\boldsymbol{SO}(2n-1)) \otimes Z_p$.

On sait que, si p est impair, $H^*(\boldsymbol{W}_{4n-1}, Z_p) \approx H^*(\boldsymbol{S}_{4n-1}, Z_p)$; le raisonnement de la Prop. 16. 1 s'applique donc sans changement et la première partie de la Proposition résulte de 13. 4.

On sait (*voir* [13], Chap. IV, Prop. 2), qu'il existe une application g de $\boldsymbol{S}_{4n-1}$ dans $\boldsymbol{W}_{4n-1}$ telle que le noyau et le conoyau des applications

$$g_0 : \pi_i(\boldsymbol{S}_{4n-1}) \to \pi_i(\boldsymbol{W}_{4n-1})$$

soient des 2-groupes pour toute valeur de i. Soit $\boldsymbol{E}$ l'espace fibré image réciproque de $\boldsymbol{SO}(2n+1)$ par cette application, et $\bar{g}$ son application canonique dans $\boldsymbol{SO}(2n+1)$. Comme la restriction de $\bar{g}$ à une fibre est un homéomorphisme sur une fibre de $\boldsymbol{SO}(2n+1)$, et comme g^* est pour tout $p \neq 2$ un isomorphisme de $H^*(\boldsymbol{S}_{4n-1}, Z_p)$ sur $H^*(\boldsymbol{W}_{4n-1}, Z_p)$, l'homomorphisme $\bar{g}^*$ est un isomorphisme de $H^*(\boldsymbol{SO}(2n+1), Z_p)$ sur $H^*(\boldsymbol{E}, Z_p)$, ([2], § 4d) ; on peut donc appliquer à $\boldsymbol{E}$ les mêmes raisonnements et calculs qu'à la fibration (c) de 17. 2. Par conséquent l'image de

$$d' : \pi_{4n-1}(\boldsymbol{S}_{4n-1}) \to \pi_{4n-2}(\boldsymbol{SO}(2n-1)) \otimes Z_p$$

est un sous-groupe de $\pi_{4n-2}(\boldsymbol{SO}(2n-1)) \otimes Z_p$ isomorphe à Z_p, ce qui, joint à l'égalité $d' = d \circ g_0$, démontre la deuxième partie de 17. 3.

18. Sur les *p*-composants des groupes d'homotopie des groupes classiques. Les Propositions précédentes permettent de calculer les p-composants des groupes $\pi_i(G)$, $(G = \boldsymbol{SU}(n), \boldsymbol{Sp}(n), \boldsymbol{SO}(n))$, jusqu'à la dimension $4p-3$. Pour énoncer les résultats obtenus, il sera commode d'utiliser le langage de la C-théorie de [13], et en particulier la notion de C-isomorphisme définie dans le Chap. I de [13]. Nous désignons par C_p la classe des groupes finis d'ordre premier à p.

Proposition 18. 1. *Soient* p *un nombre premier,* $\boldsymbol{G} = \boldsymbol{SU}(n)$. *A un* C_p*-isomorphisme près, les groupes* $\pi_i(\boldsymbol{G})$, $(i \leq 4p-3)$, *sont les suivants:*

(I). *Si* $n \leq p$, *on a* $\pi_{2j-1}(\boldsymbol{G}) \equiv Z$, $(2 \leq j \leq n)$, $\pi_{2k}(\boldsymbol{G}) \equiv Z_p$, $(p \leq k \leq p+n-2)$, $\pi_{4p-3}(\boldsymbol{G}) \equiv Z_p$, $\pi_i(\boldsymbol{G}) \equiv 0$ *sinon.*

(II). *Si* $p < n \leq 2p-2$, *on a* $\pi_{2j-1}(\boldsymbol{G}) \equiv Z$, $(2 \leq j \leq n)$, $\pi_{2k}(\boldsymbol{G}) \equiv Z_p$, $(n \leq k \leq 2p-2)$, $\pi_i(\boldsymbol{G}) \equiv 0$ *sinon.*

(III). *Si* $n \geq 2p-1$, *on a* $\pi_{2j-1}(\boldsymbol{G}) \equiv Z$, $(2 \leq j \leq 2p-1)$, $\pi_i(\boldsymbol{G}) \equiv 0$ *sinon.*

Si $n \leq p$, il résulte de [13], Chap. V, Prop. 6 que p est *régulier* pour $\boldsymbol{G}$, (au sens du § 4 de cet article), et il s'ensuit que $\pi_i(\boldsymbol{G})$ est C_p-isomorphe à la somme directe des groupes $\pi_i(\boldsymbol{S}_{2m-1})$, $(2 \leq m \leq n)$, ce qui démontre (I).

A partir de là, on raisonne par récurrence sur n pour établir (II) et (III). Il suffit pour cela de considérer la suite exacte:

$$\pi_{i+1}(\boldsymbol{S}_{2n-1}) \to \pi_i(\boldsymbol{SU}(n-1)) \to \pi_i(\boldsymbol{SU}(n)) \to \pi_i(\boldsymbol{S}_{2n-1}),$$

en remarquant que les groupes $\pi_i(\boldsymbol{S}_{2n-1})$ sont tous C_p-nuls sauf pour $i = 2n-1$ puisque $2n-1+2p-3 \geq 4p-2$ et en utilisant la Proposition 17.2 pour déterminer l'image de $d: \pi_{2n-1}(\boldsymbol{S}_{2n-1}) \to \pi_{2n-2}(\boldsymbol{SU}(n-1))$ quand $p < n \leq 2p-2$.

Le lecteur n'aura pas de peine à obtenir des résultats analogues pour $G = \boldsymbol{Sp}(n), \boldsymbol{SO}(2n+1)$ que nous n'expliciterons pas. Nous nous bornerons à indiquer comment l'on passe de $\boldsymbol{SO}(2n-1)$ à $\boldsymbol{SO}(2n)$:

PROPOSITION 18.2. *Si C désigne la classe des 2-groupes, le groupe $\pi_i(\boldsymbol{SO}(2n))$ est C-isomorphe à la somme directe de $\pi_i(\boldsymbol{SO}(2n-1))$ et de $\pi_i(\boldsymbol{S}_{2n-1})$.*

On sait que la classe caractéristique α de la fibration $\boldsymbol{SO}(2n)/\boldsymbol{SO}(2n-1) = \boldsymbol{S}_{2n-1}$ vérifie $2 \cdot \alpha = 0$. L'espace fibré $\boldsymbol{E}$, image réciproques de $\boldsymbol{SO}(2n)$ par une application de $\boldsymbol{S}_{2n-1}$ sur $\boldsymbol{S}_{2n-1}$ de degré deux, est donc isomorphe à $\boldsymbol{SO}(2n-1) \times \boldsymbol{S}_{2n-1}$. En outre, on tire de [2], § 4d, exactement comme dans la démonstration de 17.3, que l'application canonique de $\boldsymbol{E}$ sur $\boldsymbol{SO}(2n)$ définit un isomorphisme de $H^*(\boldsymbol{SO}(2n), Z_p)$ sur $H^*(\boldsymbol{E}, Z_p)$, pour tout p premier impair. Si $\bar{\boldsymbol{E}}$ et $\boldsymbol{Spin}(2n)$ sont les revêtements universels (à deux feuillets) de $\boldsymbol{E}$ et $\boldsymbol{SO}(2n)$, il en est alors de même de l'application correspondante de $H^*(\boldsymbol{Spin}(2n), Z_p)$ dans $H^*(\bar{\boldsymbol{E}}, Z_p)$; le Théorème 3 du Chap. III de [13] montre alors que $\pi_i(\bar{\boldsymbol{E}}) \to \pi_i(\mathbf{Spin}(2n))$, donc aussi $\pi_i(\boldsymbol{E}) \to \pi_i(\boldsymbol{SO}(2n))$, est un C-isomorphisme sur, ce qui termine la démonstration.

19. Groupes d'homotopie de dimension 6 des groupes classiques. Nous supposerons connues les valeurs des cinq premiers groupes d'homotopie des groupes classiques (*voir* [14], 24.11, 25.4, 25.5, et [9], 3.72) et le fait que $\pi_6(\boldsymbol{S}_3) \approx Z_{12}$.[7] Pour déterminer les 6-ièmes groupes d'homotopie, nous nous appuyerons sur la:

[7] Il est classique que $\pi_6(S_3)$ a 12 éléments; on en trouvera une démonstration simple dans [13], Chap. IV, Prop. 10. Pour prouver qu'il est cyclique, on peut, soit montrer qu'il contient un sous-groupe isomorphe à Z_4, ce qui a été fait par M. G. Barratt et G. F. Paechter, *Proc. Nat. Acad. Sci. U. S. A.*, tome 38 (1952), pp. 119-121, soit utiliser

PROPOSITION 19.1. *La classe caractéristique* $\alpha \in \pi_6(\boldsymbol{S}_3)$ *de la fibration* $\boldsymbol{Sp}(2)/\boldsymbol{Sp}(1)$ *est un générateur de* $\pi_6(\boldsymbol{S}_3)$.

Nous savons déjà par 17.2(c) que α n'est pas divisible par trois; pour obtenir 19.1, il nous suffit donc de montrer que α n'est pas divisible par deux.

Identifions $\boldsymbol{Sp}(1)$ à la sphère unité du corps des quaternions, et $\boldsymbol{S}_6$ à la variété des couples (q, q') de quaternions tels que $|q|^2 + |q'|^2 = 1$ et que la partie réelle de q' soit nulle; d'après [14], 24.11, la classe α est alors définie par l'application $g : \boldsymbol{S}_6 \to \boldsymbol{S}_3$ qui vérifie:

$$g(q, q') = 1 - 2q \cdot (1 + q')^{-2} \cdot \bar{q}.$$

D'après G. W. Whitehead [17], l'application g est homotope à g':

$$g'(q, q') = 1 - 2\,|q|^2 + 2 \cdot \frac{q \cdot q' \cdot \bar{q}}{|q|}.$$

Si l'on pose $q = Q \cdot \cos\theta$, $q' = Q' \cdot \sin\theta$, avec $0 \leq \theta \leq \pi/2$ et $|Q| = |Q'| = 1$, on voit que

$$g'(Q, Q') = -\cos 2\theta + \sin 2\theta \cdot Q \cdot Q' \cdot \bar{Q}.$$

Cela signifie que $g' : \boldsymbol{S}_6 \to \boldsymbol{S}_3$ est obtenue en faisant la construction de Hopf sur l'application de $\boldsymbol{S}_3 \times \boldsymbol{S}_2$ dans $\boldsymbol{S}_2$ donnée par $(Q, Q') \to Q \cdot Q' \cdot \bar{Q}$. Mais alors, d'après Blakers-Massey, (*Proc. Nat. Acad. Sci., U. S. A.*, vol. 35, (1949), 322-328), l'image de α par l'invariant de Hopf généralisé $H : \pi_6(\boldsymbol{S}_3) \to Z_2$ est *non nulle*; α n'est donc pas divisible par deux.

PROPOSITION 19.2. *On a* $\pi_6(\boldsymbol{Sp}(1)) = Z_{12}$, *et* $\pi_6(\boldsymbol{Sp}(n)) = 0$ *pour* $n \geq 2$.

La première égalité résulte de $\boldsymbol{Sp}(1) = \boldsymbol{S}_3$. La fibration $\boldsymbol{Sp}(2)/\boldsymbol{Sp}(1) = \boldsymbol{S}_7$ donne lieu à la suite exacte:

$$\pi_7(\boldsymbol{S}_7) \xrightarrow{d} \pi_6(\boldsymbol{S}_3) \to \pi_6(\boldsymbol{Sp}(2)) \to 0.$$

L'image de d étant le sous-groupe engendré par α est *tout* $\pi_6(\boldsymbol{S}_3)$ d'après 19.1, il s'ensuit que $\pi_6(\boldsymbol{Sp}(2)) = 0$, d'où $\pi_6(\boldsymbol{Sp}(n)) = 0$ pour $n \geq 2$.

PROPOSITION 19.3. *On a* $\pi_6(\boldsymbol{SO}(3)) = Z_{12}$, $\pi_6(\boldsymbol{SO}(4)) = Z_{12} + Z_{12}$, $\pi_6(\boldsymbol{SO}(n)) = 0$ *pour* $n \geq 5$.

$\boldsymbol{SO}(3)$ et $\boldsymbol{SO}(4)$ ont pour revêtements universels $\boldsymbol{S}_3$ et $\boldsymbol{S}_3 \times \boldsymbol{S}_3$ respectivement, d'où les deux premiers résultats. Il est classique que le revêtement universel de $\boldsymbol{SO}(5)$ est isomorphe à $\boldsymbol{Sp}(2)$, donc, vu 19.1, $\pi_6(\boldsymbol{SO}(5)) = 0$.

La fibration $\boldsymbol{SO}(6)/\boldsymbol{SO}(5) = \boldsymbol{S}_5$ donne lieu à la suite exacte:

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la détermination des groupes d'Eilenberg-MacLane en cohomologie modulo 2 due à l'un de nous, *C. R. Acad. Sci. Paris*, tome 234 (1952), pp. 1243-1245, ainsi qu'un article à paraître aux *Comm. Math. Helv.*

$$0 \to \pi_6(\boldsymbol{SO}(6)) \to \pi_6(\boldsymbol{S}_5) \xrightarrow{d} \pi_5(\boldsymbol{SO}(5)) \to \pi_5(\boldsymbol{SO}(6)).$$

Comme $\pi_5(\boldsymbol{SO}(6)) = Z$ et $\pi_5(\boldsymbol{SO}(5)) = Z_2$, (*voir* [9]), on voit que d est un isomorphisme de $\pi_6(\boldsymbol{S}_5) = Z_2$ sur $\pi_5(\boldsymbol{SO}(5))$, d'où $\pi_6(\boldsymbol{SO}(6)) = 0$.

Considérons maintenant la suite exacte:

$$0 \to \pi_6(\boldsymbol{SO}(7)) \to \pi_6(\boldsymbol{S}_6) \xrightarrow{d} \pi_5(\boldsymbol{SO}(6)) \to \pi_5(\boldsymbol{SO}(7)).$$

D'après [9], on a $\pi_5(\boldsymbol{SO}(7)) = 0$ et $\pi_5(\boldsymbol{SO}(6)) = Z$; par suite, d est un isomorphisme sur et $\pi_6(\boldsymbol{SO}(7)) = 0$; comme $\pi_6(\boldsymbol{SO}(n)) = \pi_6(\boldsymbol{SO}(7))$ pour $n \geq 7$, la démonstration de 19. 3 est achevée.

PROPOSITION 19. 4. *On a*

$$\pi_6(\boldsymbol{SU}(2)) = Z_{12},\ \pi_6(\boldsymbol{SU}(3)) = Z_6,\ \pi_6(\boldsymbol{SU}(n)) = 0 \qquad (n \geq 4).$$

La première égalité résulte de $\boldsymbol{SU}(2) = \boldsymbol{S}_3$. Examinons le groupe $\boldsymbol{SU}(3)$; la fibration $\boldsymbol{SU}(3)/\boldsymbol{S}_3 = \boldsymbol{S}_5$ donne lieu à la suite exacte:

$$\pi_7(\boldsymbol{S}_5) \xrightarrow{d} \pi_6(\boldsymbol{S}_3) \to \pi_6(\boldsymbol{SU}(3)) \to \pi_6(\boldsymbol{S}_5) \xrightarrow{d} \pi_5(\boldsymbol{S}_3) \to \pi_5(\boldsymbol{SU}(3)).$$

Montrons tout d'abord que l'homomorphisme $d: \pi_6(\boldsymbol{S}_5) \to \pi_5(\boldsymbol{S}_3)$ applique le premier groupe *sur* le second. On sait [11] que $\pi_4(\boldsymbol{SU}(3)) = 0$, donc qu'une application quelconque $\boldsymbol{S}_4 \to \boldsymbol{S}_3 \to \boldsymbol{SU}(3)$ est inessentielle; il en est a fortiori de même pour sa composée avec une application $\boldsymbol{S}_5 \to \boldsymbol{S}_4$; comme $\boldsymbol{S}_5 \to \boldsymbol{S}_4 \to \boldsymbol{S}_3$ est essentielle lorsque $\boldsymbol{S}_5 \to \boldsymbol{S}_4$ et $\boldsymbol{S}_4 \to \boldsymbol{S}_3$ le sont, cela implique que l'image de $\pi_5(\boldsymbol{S}_3)$ dans $\pi_5(\boldsymbol{SU}(3))$ est nulle, donc que d applique $\pi_6(\boldsymbol{S}_5)$ sur $\pi_5(\boldsymbol{S}_3)$. Par conséquent, la suite exacte précédente donne:

$$\pi_7(\boldsymbol{S}_5) \xrightarrow{d} \pi_6(\boldsymbol{S}_3) \xrightarrow{i} \pi_6(\boldsymbol{SU}(3)) \to 0.$$

Pour établir l'égalité $\pi_6(\boldsymbol{SU}(3)) = Z_6$, il suffit de montrer que l'image de d n'est pas nulle, autrement dit que le noyau de i est non nul. Or Hilton a prouvé que l'application composée $\boldsymbol{S}_6 \to \boldsymbol{S}_5 \to \boldsymbol{S}_4 \to \boldsymbol{S}_3$, où chaque application est essentielle, définit un élément non nul de $\pi_6(\boldsymbol{S}_3)$, (*voir* aussi [13], Chap. IV, Prop. 10, Rem. 1), et le raisonnement fait plus haut, (qui s'applique *a fortiori* ici), montre que cet élément est dans le noyau de i.

Il est bien connu que le groupe $\boldsymbol{SU}(4)$ est isomorphe au revêtement universel du groupe $\boldsymbol{SO}(6)$; on a donc $\pi_6(\boldsymbol{SU}(4)) = 0$ d'après 19. 3 et comme $\pi_6(\boldsymbol{SU}(n)) = \pi_6(\boldsymbol{SU}(4))$ pour $n \geq 4$, la Proposition 19. 4 est complètement démontrée.

20. Les fibrations des variétés de Stiefel complexes. Soient $\boldsymbol{W}_{n,q} = \boldsymbol{U}(n)/\boldsymbol{U}(n-q)$ la variété de Stiefel complexe des q-repères orthonormaux de l'espace hermitien C^n, et $\psi_{q,r}$ la projection naturelle de $\boldsymbol{U}(n)/\boldsymbol{U}(n-q) = \boldsymbol{W}_{n,q}$ sur $\boldsymbol{U}(n)/\boldsymbol{U}(n-r) = \boldsymbol{W}_{n,r}$ $(q \geq r)$. On sait ([2], § 9) que $H^*(\boldsymbol{W}_{n,q}, Z)$

est une algèbre extérieure engendrée par des éléments de dimensions $2n-2q+1$, $2n-2q+3, \cdots, 2n-1$, appliquée biunivoquement dans $H^*(\boldsymbol{U}(n), Z)$ par $\psi^*_{n,q}$, et que

$$H^*(\boldsymbol{U}(n), Z) \approx H^*(\boldsymbol{U}(n-q), Z) \otimes H^*(\boldsymbol{W}_{n,q}, Z). \tag{20.1}$$

Plus précisément, si l'on désigne par $h_1, \cdots, h_n$ les générateurs universellement transgressifs de $H^*(\boldsymbol{U}(n), Z)$ définis au No. 4, on a:

LEMME 20.2. *L'image de $H^*(\boldsymbol{W}_{n,q}, Z)$ dans $H^*(\boldsymbol{U}(n), Z)$ par $\psi^*_{n,q}$ est la sous-algèbre de $H^*(\boldsymbol{U}(n), Z)$ engendrée par les éléments h_i, $n-q+1 \leq i \leq n$.*

Soit v_i un générateur de $H^{2n-2i+1}(\boldsymbol{W}_{n,i}, Z)$ $(1 \leq i \leq n)$. C'est un élément de dimension positive minimum de $H^*(\boldsymbol{W}_{n,i}, Z)$, par conséquent, d'après un raisonnement aisé, exposé dans [2], § 23, l'élément $\psi^*_{n,i}(v_i)$ est universellement transgressif, d'où $\psi^*_{n,i}(v_i) = m_i h_i$; mais les éléments $\psi^*_{n,i}(v_i)$ forment un système de générateurs de $H^*(\boldsymbol{U}(n), Z)$, ([2], § 9, Remarque 2), donc $m_i = \pm 1$ $(1 \leq i \leq n)$.

La relation de transitivité évidente $\psi^*_{n,i} = \psi^*_{n,q} \circ \psi^*_{q,i}$, $(1 \leq i \leq q)$, montre alors que les éléments h_i $(n-q+1 \leq i \leq n)$ sont dans l'image de $\psi^*_{n,q}$; ils engendrent forcément toute cette image d'après 20.1.

Ce lemme permet de ramener le calcul des puissances réduites dans $H^*(\boldsymbol{W}_{n,q})$ au calcul analogue dans $H^*(\boldsymbol{U}(n))$, que nous avons déjà fait, (*voir* 11.4). Nous voulons en tirer des conditions nécessaires pour l'existence d'une section dans la fibration de $\boldsymbol{W}_{n,r+s}$ par $\boldsymbol{W}_{n-r,s}$, de base $\boldsymbol{W}_{n,r}$.

PROPOSITION 20.3. *Si la fibration $\boldsymbol{W}_{n,r+s}/\boldsymbol{W}_{n-r,s} = \boldsymbol{W}_{n,r}$ admet une section, on a, pour tout p premier, $\mathcal{P}_p^k(h_i) = 0$, ou, ce qui revient au même, $b_p^{k,j} \equiv 0 \bmod p$, lorsque i vérifie les inégalités:*

(1) $n-r-s < i \leq n-r$,

(2) $n-r < j \leq n$, *en posant* $j = i + k(p-1)$.

Soit h'_a l'élément de $H^*(\boldsymbol{W}_{n,r+s})$ vérifiant

$$\psi^*_{n,r+s}(h'_a) = h_a \qquad (n-r-s < a \leq n)$$

et soit de même $h''_b \in H^*(\boldsymbol{W}_{n,r})$ tel que

$$\psi^*_{n,r}(h''_b) = h_b \qquad (n-r < b \leq n).$$

On a évidemment $\psi^*_{r+s,r}(h''_b) = h'_b$, et si $s: \boldsymbol{W}_{n,r} \to \boldsymbol{W}_{n,r+s}$ est une section, l'égalité $s^* \circ \psi^*_{r+s,r} = 1$, donne alors:

$$s^*(h'_i) = 0 \qquad (n-r-s < i \leq n-r)$$

$$s^*(h'_j) = h''_j \qquad (n-r < j \leq n); \tag{20.4}$$

si maintenant nous supposons que i vérifie les inégalités de 20. 3, on déduit de 11. 4, 20. 2 et 20. 4 :

$$0 = \mathcal{P}_p{}^k(s^*(h'_i)) = s^*(\mathcal{P}_p{}^k(h'_i)) = s^*(b_p{}^{k,j} \cdot h'_j) = b_p{}^{k,j} \cdot s^*(h'_j) = b_p{}^{k,j} \cdot h''_j$$

donc $b_p{}^{k,j} \equiv 0 \ mod\ p$.

Corollaire 20. 5. *Si la fibration* $\boldsymbol{W}_{n,r+s}/\boldsymbol{W}_{n-r,s} = \boldsymbol{W}_{n,r}$ *admet une section, on a* $s = 1$ *ou* $r = 1$.

Supposant $r \geq 2$ et $s \geq 2$, nous appliquerons la Proposition 20. 3 avec $p = 3$; distinguons deux cas :

a) $n - r \equiv 2 \bmod 3$; on pose $i = n - r$, $k = 1$; on a donc $j = i + 2 = n - r + s$ et les inégalités (1) et (2) de 20. 3 sont vérifiées puisque $r \geq 2$; de plus, vu l'hypothèse faite sur $n - r$, on a $j \equiv 1 \bmod 3$. Mais d'après 12. 3 $b_3{}^{1,j} \equiv j \bmod 3$, donc $b_3{}^{1,j} \not\equiv 0 \bmod 3$ et il n'y a pas de section vu 20. 3.

(b) $n - r \not\equiv 2 \bmod 3$. On pose $i = n - r - 1$, $k = 1$; les inégalités (1) et (2) sont vérifiées puisque $s \geq 2$; mais $j = n - r + 1 \not\equiv 0 \bmod 3$, d'où comme précédemment, $b_3{}^{1,j} \not\equiv 0 \bmod 3$.

Nous traiterons maintenant plus en détails le cas $r = 1$; on pourrait étudier de même le cas $s = 1$, mais, les calculs étant plus compliqués, nous ne nous y attarderons pas.

Proposition 20. 6. *Si la fibration* $\boldsymbol{W}_{n,s+1}/\boldsymbol{W}_{n-1,s} = \boldsymbol{W}_{n,1} = \boldsymbol{S}_{2n-1}$ *admet une section, n est divisible par l'entier*

$$N_s = \Pi_p \, p^{1+h(p,s)}$$

où le produit est étendu à les nombres premiers, et où $h(p, s)$ *désigne pour tout p le plus grand entier* $h \geq -1$ *tel que* $(p-1) \cdot p^h \leq s$.

Vu la Proposition 20. 3, il nous suffira de prouver ceci :

Lemme 20. 7. *Soient p un nombre premier, s un entier* $< n$, *et supposons que* $b_p{}^{k,n} \equiv 0 \ mod\ p$ *pour tout entier* $k > 0$ *tel que* $n - k(p-1) \geq n - s$, *(autrement dit tel que* $k(p-1) \leq s$). *Alors n est divisible par* $p^{1+h(p,s)}$.

Ce lemme sera lui-même conséquence du lemme suivant, dans lequel s n'intervient plus :

Lemme 20. 8. *Soient a un entier* ≥ 0, $k = p^a$, *supposons* $n > k(p-1)$ *et n divisible par k. Si* $b_p{}^{k,n} \equiv 0 \ mod\ p$, *alors n est divisible par* p^{a+1}.

Montrons, par récurrence sur s, que 20. 8 entraîne 20. 7. Ce dernier est trivial pour $s = 0$, supposons le vrai pour $s - 1$; puisque

$$h(p, s) \leq 1 + h(p, s-1),$$

cela implique que $p^{h(p,s)}$ divise n; mais $(p-1)\cdot p^{h(p,s)} \leq s < n$ par définition de $h(p,s)$, par conséquent l'entier $a = p^{h(p,s)}$ vérifie les hypothèses de 20.8 et n est bien divisible par $p^{h(p,s)+1}$.

Le lemme 20.8 est un cas particulier du résultat suivant, que nous allons maintenant démontrer:

LEMME 20.9. *Soient F le polynôme symétrique $\Sigma\, x_1^{\alpha_1}\cdot\cdots\cdot x_i^{\alpha_i}$, de degré $n = \Sigma\,\alpha_i$, p un nombre premier, a un entier. On suppose que p^a divise n et qu'il y a au plus p^a indices j tels que $\alpha_j \neq 1$.*

Dans ces conditions, pour que le terme dominant de F soit $\not\equiv 0 \bmod p$, il faut et il suffit, ou bien que $\alpha_j = 1$ pour tout j, ou bien qu'il y ait exactement p^a indices j tels que $\alpha_j \neq 1$, les α_j correspondants étant tous égaux et n n'étant pas divisible par p^{a+1}.

(On obtient 20.8 en appliquant 20.9 au cas où $\alpha_j = p$ pour $1 \leq j \leq p^a$ et $\alpha_j = 1$ pour $j > p^a$.)

Nous démontrerons le lemme 20.9 par récurrence descendante sur i, le cas $i = n$ étant trivial puisque tous les α_j sont alors égaux à 1.

Nous supposerons que l'on a $\alpha_1, \cdots, \alpha_k > 1$, et $\alpha_{k+1} = \cdots = \alpha_i = 1$. On a $1 \leq k \leq p^a$, vu $i < n$. Considérons le polynôme symétrique:

$$G = (\Sigma\, x_1^{\alpha_1 - 1}\cdots x_k^{\alpha_k - 1})\cdot(\Sigma\, x_1\cdots x_i)\,;$$

dans le développement de G, nous trouverons évidemment le polynôme F; quant aux autres termes ce seront des polynômes symétriques de la forme:

$$c_{(\beta)}\Sigma\, x_1^{\beta_1}\cdots x_k^{\beta_k}\cdot x_{k+1}\cdots x_{i'},$$

où $i' = n + k - \Sigma\,\beta_j$, et où β_j est égal soit à α_j, soit à $\alpha_j - 1$, ce second cas se présentant pour au moins une valeur de i, (ce qui montre que $i' > i$); le coefficient $c_{(\beta)}$ est un entier.

G est le produit de deux polynômes symétriques, il est donc décomposable et l'égalité:

$$G = F + \Sigma_{(\beta)} c_{(\beta)}\Sigma\, x_1^{\beta_1}\cdots x_k^{\beta_k}\cdot x_{k+1}\cdots x_{i'},$$

montre que le terme dominant de F, changé de signe, est égal à la somme des termes dominants des polynômes $c_{(\beta)}\Sigma\, x_1^{\beta_1}\cdots x_k^{\beta_k} x_{k+1}\cdots x_{i'}$; comme $i' > i$, on peut appliquer le lemme à ces derniers vu l'hypothèse de récurrence. Nous distinguerons quatre cas:

(A) *On a $\alpha_j = 2$ pour $j \leq k$.* Dans ce cas, le seul choix des β_j qui conduise à un terme dominant non nul est celui où $\beta_j = 1$ pour tout j; il est immédiat que le coefficient $c_{(\beta)}$ correspondant est égal au coefficient binomial $\binom{n}{k}$, et, puisque $1 \leq k \leq p^a$ et que p^a divise n, les propriétés de divisibilité des coefficients binomiaux montrent que:

$$c_{(\beta)} = \binom{n}{k} \equiv 0 \bmod p \quad \text{si } k < p^a,$$

$$c_{(\beta)} = \binom{n}{k} \equiv n/p^a \bmod p \quad \text{si } k = p^a,$$

ce qui établit le lemme sous l'hypothèse (A).

(B) *On a* $\alpha_j = q > 2$ *pour* $j \leq k$. Le seul choix des β_j qui conduise à un terme dominant non nul est celui où $\beta_j = q - 1$ pour $1 \leq j \leq k$, lorsque de plus $k = p^a$ et p^{a+1} ne divise pas n; on voit alors tout de suite que $c_{(\beta)} = 1$, ce qui démontre le lemme dans le cas (B).

(C) *On a* $k < p^a$. Le seul choix des β_j qui conduise à un terme dominant non nul est celui où $\beta_j = 1$ pour tout j; mais les α_j, $(1 \leq j \leq k)$, sont alors égaux à 2, et l'on se retrouve dans le cas (A) déjà traité.

(D) *On a* $k = p^a$, *et les* α_j, $(1 \leq j \leq k)$, *ne sont pas tous égaux.* Dans ce cas, le seul choix des β_j qui conduise à un terme dominant non nul est $\beta_1 = \cdots = \beta_k = q > 1$, et ce choix n'est possible que si certains des α_j sont égaux à $q + 1$, et tous les autres à q; soient r et s leurs nombres respectifs. Vu l'hypothèse faite on a $r + s = p^a$, $r \geq 1$, $s \geq 1$; comme $q > 1$, il est immédiat que

$$c_{(\beta)} = \binom{r+s}{r} = \binom{p^a}{r} \equiv 0 \bmod p,$$

d'où le lemme dans le cas (D).

Comme les cas (A), (B), (C), (D) épuisent toutes les possibilités, la démonstration du lemme 20. 8 est achevée, et la Proposition 20. 6 est complètement établie.

Exemples.

$s = 1$; pour $p = 2$, on a $h(p, 1) = 0$, pour $p > 2$, $h(p, 1) = -1$, d'où $N_1 = 2$. Dans ce cas du reste, la condition " n pair " est non seulement nécessaire mais suffisante pour l'existence d'une section (*voir* Eckmann, [8], Satz IV).

$s = 2$; pour $p = 2$, $h(p, 2) = 1$, pour $p = 3$, $h(p, 2) = 0$, pour $p \geq 5$, $h(p, 2) = -1$, donc $N_s = 12$. Nous ignorons si la condition " n divisible par 12 " est suffisante pour l'existence d'une section, mais cela semble peu probable. Il est assez naturel de conjecturer que $\boldsymbol{W}_{n,s+1}/\boldsymbol{W}_{n-1,s} = \boldsymbol{S}_{2n-1}$ n'a pas de section si $s > 1$.

Indiquons pour terminer quelques valeurs de la fonction arithmétique N_s:

$$N_1 = 2,\ N_2 = N_3 = 12,\ N_4 = N_5 = 120,\ N_6 = N_7 = 2.520, \cdots,$$

$$N_{20} = N_{21} = 6.983.776.800.$$

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ON THE LOCAL BEHAVIOR OF SOLUTIONS OF NON-PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS.*

By PHILIP HARTMAN and AUREL WINTNER.

This paper deals with local problems concerning binary elliptic systems of partial differential equations in two independent variables and with corresponding partial elliptic differential equations of second order. The results are known in the analytic case, but most of them are not known even in the C^∞-case. The transition to non-analytic cases will be made possible by appropriate applications of the ideas contained in a short paper of Carleman [1]. At the same time, Carleman's results will be made more precise.

At the end of the paper, corresponding extensions and applications will be made of the results of Carleman [2] on non-parabolic systems in n dependent and two independent variables.

1. The gradient. For a function u of (x, y), let p, q and r, s, t denote, as usual, the first and second order partial derivatives. In what follows, all partial differential equations are assumed to involve only real-valued coefficients and/or functions, and all solutions are assumed to be real-valued, unless the contrary is said or implied.

THEOREM 1. *Let $d(x, y)$, $e(x, y)$, $f(x, y)$ be continuous functions on a circle*

$$x^2 + y^2 < R^2. \tag{1}$$

Let $u = u(x, y)$ be a function, of class C^2 on (1), satisfying the partial differential equation

$$r + t + dp + eq + fu = 0 \tag{2}$$

or, more generally, let $u(x, y)$ be a function, of class C^1 on (1), satisfying the integral relation

$$\int_J q\,dx - p\,dy = \iint_E (dp + eq + fu)\,dx\,dy \tag{3}$$

for every domain E bounded by a piecewise smooth (C^1) Jordan curve J

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* Received January 9, 1953.

contained in (1). *Then, if the solution* u *behaves, as* $(x, y) \to (0, 0)$, *in such a way that*

$$u(x, y) = o(\rho^n), \quad \text{where} \quad \rho = (x^2 + y^2)^{\frac{1}{2}}, \tag{4}$$

holds for some non-negative integer n, *the gradient* (u_x, u_y) *of* u *must behave in such a way that*

$$\lim_{\rho \to 0} (u_y + iu_x)/(x + iy)^n \text{ exists} \tag{5}$$

(*for the same* $n \geqq 0$).

If the limit in (5) is denoted by $c_2 + ic_1$, where c_1, c_2 are real constants, then the assertion (5) can be written as

$$\begin{aligned} u_x &= \rho^n(c_1 \cos n\theta + c_2 \sin n\theta) + o(\rho^n), \\ u_y &= \rho^n(c_2 \cos n\theta - c_1 \sin n\theta) + o(\rho^n), \end{aligned} \tag{5'}$$

where $x = \rho \cos \theta$, $y = \rho \sin \theta$, which implies that the assumption (4) can be refined to

$$u = (n+1)^{-1}\rho^{n+1}\{c_1 \cos(n+1)\theta + c_2 \sin(n+1)\theta\} + o(\rho^{n+1}). \tag{5''}$$

In fact, (5″) follows if (5′) is inserted into the identity

$$u = \int_0^\rho (p \cos \theta + q \sin \theta)\, d\rho,$$

in which the argument of u, p, q is $(\rho \cos \theta, \rho \sin \theta)$.

The content of the following theorem is that, unless $u \equiv 0$, the relation (4) cannot hold for every n; so that (5′), (5″) hold for some $n \geqq 0$ with constants c_1, c_2 which are not both 0.

THEOREM 2. *In addition to the conditions of Theorem* 1, *assume that* $u \not\equiv 0$. *Then there exists a least integer* $m\,(> 0)$ *such that* (4) *fails to hold for* $n = m$ (*and, therefore, the limit in* (5) *is not* 0 *when* $n = m - 1$).

An immediate consequence of Theorems 1 and 2 is the following:

COROLLARY 1. *Under the assumptions of Theorems* 1 *and* 2, *the zeros of* grad $u = (u_x, u_y)$ *cannot cluster at* $(x, y) = (0, 0)$.

REMARK. It will be clear from the proofs that Theorems 1 and 2 remain unaltered if $dp + eq + fu$ in (2) or (3) is replaced by a more general function of (x, y, u, p, q), say $g = g(x, y, u, p, q)$, which is continuous in its

five variables and is subject to the restriction that, for every $\epsilon > 0$ and every $M > 0$, there exists a constant $K = K(\epsilon, M)$ satisfying

$$| g(x, y, u, p, q) | \leqq K \cdot (| u | + | p | + | q |)$$

when $x^2 + y^2 < \epsilon^2$ and $| u | \leqq M$, $| p | \leqq M$, $| q | \leqq M$. Similar remarks apply to the theorems below dealing with linear differential equations.

It is seen from (3) and the integrability condition

$$\int p dx + q dy = 0 \tag{6}$$

that, if $f \equiv 0$ and if u is of class C^2, then the functions p, q are of class C^1 and satisfy a system of partial differential equations of the type considered by Carleman [1]. Hence, in this case, the first (non-parenthetical) part of Theorem 2 and Corollary 1 are contained in Carleman's results. On the other hand, Theorem 1 and the last (parenthetical) part of Theorem 2 go beyond the results of Carleman and supply the asymptotic formulae (5′), (5″).

In the note referred to, Carleman states, without proof, a theorem on non-linear elliptic equations which implies (whether or not $f \equiv 0$) a weaker conclusion than that of Corollary 1, namely, that the *common zeros of* grad u *and of* u cannot cluster at $(x, y) = (0, 0)$. It turns out that Theorems 1 and 2 imply that the conclusion of this theorem of Carleman can be strengthened along the lines of the conclusion of Corollary 1; cf. Section **12** below.

2. Counterexamples. It is instructive to contrast the cases $f \equiv 0$ and $f \not\equiv 0$. If $f \equiv 0$, then the assumption (4) can be replaced by

$$u(x, y) = u(0, 0) + o(\rho^n) \tag{7}$$

without invalidating the conclusion (5) of Theorem 1, the corresponding conclusion of Theorem 2 (namely, that (7) cannot hold for every n unless $u \equiv u(0, 0)$) and the assertion of Corollary 1. If, on the other hand, $f \not\equiv 0$ and $u(0, 0) \neq 0$, then none of these conclusions holds; in other words, it is not true that (7) implies (5), nor is it true that $u \equiv$ const. if (7) holds for every n; finally, it is not true that $u \equiv$ const. when the zeros of grad u cluster at $(x, y) = (0, 0)$. With regard to the last negative assertion, the situation is the same as in the case of an *ordinary*, homogeneous linear differential equation

$$u_{xx} + f(x)u = 0, \qquad u = u(x). \tag{8}$$

The positive statements (concerning the case $f \equiv 0$) will be clear from

the proofs of Theorems 1 and 2. The negative statements (concerning the case $f \not\equiv 0$) can be verified by simple examples of (2) in which $d \equiv e \equiv 0$, and f is a function of x alone. In this case, any solution $u = u(x)$ of (8), when considered as a function of (x, y), is a solution of (2). The desired examples can be obtained by starting with a suitable function $u = u(x)$, where $u(x) \neq 0$, and defining $f(x)$ by (8). Thus, the example $u(x) = 1 + x^5 \sin(x^{-1})$, where $x \neq 0$ and $u(0) = 1$, proves the first and the third of the negative assertions (which state that the corresponding conclusions of Theorem 1 and Corollary 1 do not hold). The example $u(x) = 1 + \exp(-x^{-2})$, where $x \neq 0$ and $u(0) = 1$, proves the second negative assertion (which states that (7) can hold for every n even if $u \not\equiv$ const.).

3. A binary elliptic linear system. The proofs of Theorems 1 and 2 will imply a proof of the following theorem (*), dealing with the integral analogue of a system of differential equations treated by Carleman [1]. His result is to the effect that (10) below cannot hold for every n (when u, v are of class C^1). The conclusion of (*) goes a great deal further. The consideration of the integral relations (9_1)-(9_2) below, instead of their differentiated form

$$v_x + u_y + \alpha u + \beta v = 0, \qquad v_y - u_x + \gamma u + \delta v = 0, \tag{9 bis}$$

makes it possible to assume that u, v are only continuous.

THEOREM (*). *Let $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$, $\delta(x, y)$ be continuous functions on* (1). *Let $u = u(x, y)$, $v = v(x, y)$ be continuous functions on* (1) *satisfying*

$$\int_J u\,dx - v\,dy = \iint_E (\alpha u + \beta v)\,dx\,dy, \tag{9_1}$$

$$\int_J v\,dx + u\,dy = \iint_E (\gamma u + \delta v)\,dx\,dy \tag{9_2}$$

for every domain E bounded by a piecewise smooth (C^1) Jordan curve contained in (1) *(for instance, let u, v be of class C^1 and satisfy* (9 bis)), *and suppose that, for some non-negative integer n,*

$$u = o(\rho^n) \quad \text{and} \quad v = o(\rho^n) \tag{10_n}$$

as $\rho \to 0$, where $\rho = (x^2 + y^2)^{\frac{1}{2}}$. Then

$$\lim_{\rho \to 0} (u + iv)/(x + iy)^{n+1} \text{ exists} \tag{11_n}$$

(for the same n). Moreover, unless both u and v vanish identically, there exists a least integer $n \geqq 0$ for which the limit (11_n) *is not* 0.

Theorem (*) will be clear from the proofs of Theorems 1 and 2. In fact, (9_1)-(9_2) imply that

$$\int_J (u+iv)(dx+idy) = \iint_E \{(\alpha+i\gamma)u+(\beta+i\delta)v\}dxdy,$$

and this relation can be used in the same way as (14) is used below in the proofs of Theorems 1 and 2.

4. Removable singularities. The Green formulae to be applied in the proof of Theorem 1 can be used to prove theorems generalizing the classical facts on the removable singularities of harmonic functions. In this direction there will be proved the following theorem (which can be considered to be the case $n=-1$ of Theorem (*)):

THEOREM ($\square$). *Let the coefficient functions of a system* (9 bis) *be continuous on the circle* (1) *and let* u, v *be functions of class* C^1 *on the punctured circle* $0<x^2+y^2<R^2$, *on which they satisfy* (9 bis); *while, as* $x^2+y^2\to 0$, *let*

$$(10_{-1}) \qquad u=o(1/\rho) \quad \text{and} \quad v=o(1/\rho),$$

where $\rho^2=x^2+y^2$. *Then*

(I) u, v *can be defined at* $(x,y)=(0,0)$ *so as to render* u, v *continuous at* $(x,y)=(0,0)$; *that is, the limits*

$$(11_{-1}) \qquad \lim_{\rho\to 0} u(x,y) \quad \text{and} \quad \lim_{\rho\to 0} v(x,y) \quad \text{exist};$$

(II) *if, in addition, both limits* (11_{-1}) *are* 0 *and, correspondingly,*

$$(10_0) \qquad u(0,0)=0 \quad \text{and} \quad v(0,0)=0,$$

then the partial derivatives u_x, u_y, v_x, v_y *exist, and* (9 bis) *is satisfied, at* $(x,y)=(0,0)$.

Note that (II) does not claim the continuity of the partial derivatives at $(0,0)$; that is, the C^1-character of the solution on (1).

If the additional condition (10_0) is omitted, then the assertion of (II) is false in general. In order to see this, let $g(x,y)$ be $x^2/\{(x^2+y^2)\log(x^2+y^2)\}$ or 0 according as $\rho\neq 0$ or $\rho=0$. Then, according to Petrini [7], p. 138, the Poisson equation $z_{xx}+z_{yy}=g$ has no solution of class C^2 on the circle $x^2+y^2<R^2$ (for any $R<1$). But there exist functions $z=z(x,y)$ which are of class C^1 on the circle (1), are of class C^2 and satisfy the Poisson equation on the punctured circle $0<x^2+y^2<R^2$, while $z_{xx}(0,0)$ and

$z_{yy}(0,0)$ fail to exist. Since a harmonic function can be added to z, it can be supposed that $z_x(0,0) \neq 0$, hence, if $R > 0$ is small enough, $z_x \neq 0$ on (1). Then $u = z_y(x,y)$ and $v = z_x(x,y)$ are continuous on (1) and are of class C^1 and satisfy (9 bis) for $0 < x^2 + y^2 < R^2$, where $\alpha \equiv 0$, $\beta = -g(x,y)/z_x(x,y)$ and $\gamma \equiv \delta \equiv 0$. Nevertheless, $u_y(0,0)$ and $v_x(0,0)$ fail to exist.

On the other hand, the additional condition (10_0) can be omitted in (II) if additional conditions are imposed on the coefficient functions α, β, γ, δ:

THEOREM ($\square$ bis). *Let the coefficient functions of* (9 bis) *satisfy a uniform Hölder condition (of some order* λ, *where* $0 < \lambda < 1$) *in the circle* (1), *and let* u, v *satisfy the conditions of part* (I) *of Theorem* ($\square$). *Then* u, v *are of class* C^1 *on* (1) (and their partial derivatives u_x, u_y, v_x, v_y satisfy, on every compact subset of (1), a uniform Hölder condition of every order $\mu < \lambda$).

It will be clear from the proof that, in Theorems ($\square$) and ($\square$ bis), the condition that u, v are of class C^1 and satisfy (9 bis) on $0 < x^2 + y^2 < R^2$ can be reduced to the assumption that u, v are continuous and satisfy (9_1), (9_2) for every simply connected subdomain E of $0 < x^2 + y^2 < R^2$, bounded by a piecewise smooth (C^1) Jordan curve contained in $0 < x^2 + y^2 < R^2$.

It will also be clear from the proofs that Theorems ($\square$) and ($\square$ bis) have analogues in which the first order system (9 bis) is replaced by the second order equation (2).

REMARK TO THEOREM ($\square$). The proof of Theorem ($\square$) can be modified so as to show that if d, e, f are continuous on (1) and if, on the punctured circle $0 < x^2 + y^2 < R^2$, the function $u = u(x,y)$ is of class C^2, satisfies (2) and has partial derivatives p, q satisfying $p^2 + q^2 = o(1/\rho)$, then u can be defined at $(x,y) = (0,0)$ so as to become of class C^1 on (1). Also, if $u(0,0) = p(0,0) = q(0,0)$, then u_{xx}, $u_{xy} = u_{yx}$, u_{yy} exist, and (2) is satisfied, at $(x,y) = (0,0)$.

REMARK TO THEOREM ($\square$ bis). The proof of Theorem ($\square$ bis) will depend on a modification of a device of Lichtenstein [12], p. 1321, which implies that if u is continuous on (1) and satisfies the integral identity (3) in $E = E(J)$, then u is of class C^1 (and its partial derivatives satisfy, on every compact subset of (1), a uniform Hölder condition of every order $\mu < 1$) when d, e, f are continuous on (1); and that u is of class C^2 (and that its second order partial derivatives satisfy, on every compact subset of (1), a

uniform Hölder condition of every order $\mu < \lambda$) when d, e, f satisfy a uniform Hölder condition of order λ on (1); cf. [14], p. 735.

5. Proof of Theorem 1. Put $z = x + iy$, use the notation $g(z)$ for any function $g(x, y)$ of (x, y) (so that $g_x(z)$, $g_y(z)$ mean $\partial g(x, y)/\partial x$, $\partial g(x, y)/\partial y$, respectively), and let

$$w = u_y + iu_x, \qquad (13) \qquad W = du_x + eu_y + fu. \tag{12}$$

Then (3) and the integrability condition (6) can be written as

$$\int_J w dz = \iint_E W dx dy. \tag{14}$$

According to the Lemma of [5], p. 761, the identity (14) in $E = E(J)$ implies that

$$\int_J gw dz = \iint_E \{gW + iw(g_x + ig_y)\} dx dy,$$

if $g = g(x, y)$ is any function of class C^1 on $E + J$. Let this be applied to the function $g(z) = z^{-k}(z - \zeta)^{-1}$, where $\zeta \neq 0$, k is a non-negative integer and $E = E_\epsilon$ is the domain bounded by the circles $|z| = R$, $|z| = \epsilon$ and $|z - \zeta| = \epsilon$ (where $\epsilon > 0$ is small and R is any fixed positive number smaller than the R in (1)). Since $g(z)$ is a regular function on E, hence $g_x + ig_y \equiv 0$, there results the identity

$$\int_J wz^{-k}(z - \zeta)^{-1} dz = \iint_E Wz^{-k}(z - \zeta)^{-1} dx dy. \tag{15}$$

If $w(z)$ satisfies

$$w = o(|z|^{k-1}) \tag{16_k}$$

as $z \to 0$, then it follows from (15), on letting $\epsilon \to 0$, that

$$2\pi i w(\zeta)\zeta^{-k} = \int_{|z|=R} wz^{-k}(z - \zeta)^{-1} dz - \iint_{|z|<R} Wz^{-k}(z - \zeta)^{-1} dx dy, \tag{17_k}$$

where the double integral is absolutely convergent.

Since Theorem 1 is trivial if $0 \leqq n \leqq 1$, it will be supposed that $n > 1$. It will be shown by an induction (on k for fixed n) that (4) implies (16_n). Clearly, $n > 1$ in (4) implies (16_1). Suppose that (16_k) holds for a k satisfying $1 \leqq k < n$. It will be verified that (16_{k+1}) holds.

The definition (13) shows that

$$|W| \leqq K(|w| + |u|) \tag{18}$$

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holds for some constant $K = K_R > 0$. Hence

$$(19_k) \qquad 2\pi \mid w(\zeta)\zeta^{-k} \mid \leqq \int_{|z|=R} \mid wz^{-k}(z-\zeta)^{-1} \mid \cdot \mid dz \mid + K \iint_{|z|<R} (\mid w \mid + \mid u \mid) \mid z \mid^{-k} \mid z-\zeta \mid^{-1} dxdy.$$

Let this inequality be multiplied by $\mid \zeta - z_0 \mid^{-1} d\xi d\eta$ and then integrated over $\mid \zeta \mid < R$, where $\zeta = \xi + i\eta$. Then, if use is made of the inequality (cf. [1], p. 473)

$$(20) \qquad \iint_{|z|<R} \mid z-\zeta \mid^{-1} dxdy < 2R, \quad \text{where} \quad \mid \zeta \mid < R,$$

and of the identity

$$(21) \qquad \mid (z-\zeta)(\zeta - z_0) \mid^{-1} = \mid z - z_0 \mid^{-1} \mid (z-\zeta)^{-1} + (\zeta - z_0)^{-1} \mid,$$

it follows that

$$2\pi \iint_{|z|<R} \mid w(z)z^{-k}(z-\zeta)^{-1} \mid dxdy \leqq 4R \int_{|z|=R} \mid wz^{-k}(z-\zeta)^{-1} \mid \cdot \mid dz \mid + 4KR \iint_{|z|<R} (\mid w \mid + \mid u \mid) \mid z \mid^{-k} \mid z-\zeta \mid^{-1} dxdy,$$

where ζ has been written in place of z_0. Accordingly,

$$(2\pi - 4KR) \iint_{|z|<R} \mid wz^{-k}(z-\zeta)^{-1} \mid dxdy \leqq 4R \int_{|z|=R} \mid wz^{-k}(z-\zeta)^{-1} \mid \cdot \mid dz \mid + 4KR \iint_{|z|<R} \mid uz^{-k}(z-\zeta)^{-1} \mid dxdy.$$

It can be supposed that R is so small that $2\pi - 4KR > 0$. Since $k < n$, hence

$$(22) \qquad uz^{-k} = O(1)$$

as $z \to 0$, the last double integral is $O(1)$ as $\zeta \to 0$. It follows therefore from (19_k) that $w(\zeta)\zeta^{-k} = O(1)$.

Thus, $W(z)z^{-k} = O(1)$ as $z \to 0$. This implies that the double integral in (17_k) tends to a limit as $\zeta \to 0$ (in fact, the absolute continuity of the set function $\iint_E Wz^{-k}(z-\zeta)^{-1} dxdy$ of E is uniform with respect to ζ, as $\zeta \to 0$; cf. (20)). Hence, by (17_k),

$$(23_k) \qquad \lim_{z \to 0} w(z)z^{-k} \text{ exists.}$$

If $k < n$, then the limit (23_k) is 0; for otherwise $u \neq o(|z|^{k+1})$, which contradicts (4), since $k+1 \leqq n$. Since this completes the induction, (16_n) holds for every n.

On the other hand, (23_k) was deduced as a consequence of (16_k) and (22). Hence (23_n), that is, (5), is true. This proves Theorem 1.

6. Proof of Theorem 2. Suppose that (4) holds for every positive integer n. It will be shown that $u(z) \equiv 0$ on (1). According to the proof of Theorem 1, (4) implies (16_n), which in turn implies (19_n). In view of (20), an integration of (19_n) leads to the inequality

$$2\pi \iint_{|z|<R} |wz^{-n}|\,dxdy \leqq 2R \int_{|z|=R} |wz^{-n}| \cdot |dz| + 2KR \iint_{|z|<R} (|w| + |u|)|z|^{-n}dxdy,$$

which can be written as

$$(24) \qquad (2\pi - 2KR) \iint_{|z|<R} |wz^{-n}|\,dxdy \leqq 2R \int_{|z|=R} |wz^{-n}| \cdot |dz| + 2KR \iint_{|z|<R} |u|\,|z|^{-n}dxdy$$

(this inequality corresponds to (4) in [1], p. 473).

In order to appraise the last integral, note that

$$u(z) = \int_0^1 (u_x(tz)x + u_y(tz)y)\,dt; \text{ so that } |u(z)| \leqq \int_0^1 |zw(tz)|\,dt,$$

by (12). If this inequality is multiplied by $|z|^{-n}$, an integration gives

$$\iint_{|z|<R} |uz^{-n}|\,dxdy \leqq \int_0^1 \{\iint_{|z|<R} |z^{-n+1}w(tz)|\,dxdy\}dt.$$

The last (triple) integral is transformed into

$$\int_0^1 t^{n-3}\{\iint_{|z|<tR} |z^{-n-1}w(z)|\,dxdy\}dt$$

by the change of variables $tz \to z$ in the interior double integral. If $n \geqq 3$, there results the inequality

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$$\int\limits_{|z|<R} | uz^{-n} | \, dxdy \leqq \int\int\limits_{|z|<R} | wz^{-n+1} | \, dxdy.$$

Here the factor z^{-n+1} can be replaced by z^{-n} if it is supposed that $R < 1$

Thus it follows from (24) that

$$(25) \qquad (2\pi - 4KR) \int\int\limits_{|z|<R} | wz^{-n} | \, dxdy \leqq 2R \int\limits_{|z|=R} | wz^{-n} | \cdot | \, dz \, |$$

for $n = 3, 4, \cdots$.

Let $R(>0)$ be fixed and so small that $R < 1$ and $2\pi - 4KR > 0$. Suppose, if possible, that there exists a point $z = z_0$ in $| z | < R$ for which $w(z_0) \neq 0$. It is clear that the left side of (25) exceeds const. $| z_0 |^{-n}$ for some const. > 0 which is independent of n. On the other hand, the right side of (25) is majorized by Const. R^{-n}. In view of $| z_0 | < R$, this contradicts (25) for large n. Hence $w(z) \equiv 0$, and so $u(z) \equiv u(0) = 0$, for $| z | < R$.

Accordingly, the assumption of (4) for $n = 1, 2, \cdots$ implies that $u(z) \equiv 0$ holds on a circle $| z | < R$, where R is subject only to the restriction $R < \max(1, \frac{1}{2}\pi/K)$. Consequently, $u(z) \equiv 0$ holds on (1) without any restriction on R. This proves Theorem 2.

7. Proof of (I) in Theorem ($\square$). In this proof, the letter R will denote a positive number smaller than the R in (1). Let

$$(26) \qquad w = u + iv.$$

Then $w = w(x, y) = w(z)$ is $o(| z |^{-1})$ as $z \to 0$, by (10_{-1}). Hence, if $\zeta \neq 0$,

$$(27) \qquad 2\pi i w(\zeta) = \int\limits_{|z|=R} w(z)/(z - \zeta) dz - \int\int\limits_{|z|<R} (\alpha^* u + \beta^* v)/(z - \zeta) \, dxdy,$$

where $\alpha^* = \alpha + i\gamma$ and $\beta^* = \beta + i\delta$; cf. the proof (17_k). In fact, condition (10_{-1}) implies that, in the derivation of (27), the contributions of the line integral over $| z | = \epsilon$ and of the double integral over $| z | < \epsilon$ tend to 0 as $\epsilon \to 0$ (the double integral in (27) is absolutely convergent). If $K = K(R)$ is a constant such that $| \alpha^* |$ and $| \beta^* |$ are majorized by K for $| z | < R$, then (27) shows that

$$2\pi | w(\zeta) | \leqq \int\limits_{|z|=R} | w(z)/(z - \zeta) | \cdot | \, dz \, | + K \int\int\limits_{|z|<R} | w(z)/(z - \zeta) | \, dxdy.$$

If this inequality is multiplied by $d\xi d\eta/|\zeta - z_0|$, an integration and (20), (21) lead to

$$(2\pi - 4KR)\iint_{|z|<R} |w(z)/(z-\zeta)|\,dxdy \leqq 4R\int_{|z|=R} |w(z)/(z-\zeta)|\cdot|dz|,$$

if ζ is written in place of z_0. If R is so small that $2\pi - 2KR > 0$, then the last two formula lines show that $w(\zeta)$ remains bounded as $\zeta \to 0$; cf. the proof of (16_k). Consequently, (27) implies that $w(\zeta)$ tends to a limit as $\zeta \to 0$; cf. the argument leading to (23_k). This proves (I).

Proof of (II) in Theorem ($\square$). The assumption (10_0) means that (26) satisfies $w = o(1)$ as $\zeta \to 0$. Hence w satisfies the identity

$$2\pi i w(\zeta)/\zeta = \int_{|z|=R} w(z)/(z-\zeta)dz - \iint_{|z|<R} (\alpha^* u + \beta^* v)/z(z-\zeta)dxdy;$$

cf. the proof of (17_k). The above arguments show, first, that $w(\zeta)/\zeta$ is bounded, and then, that $\lim w(\zeta)/\zeta$ exists, as $\zeta \to 0$. Hence the functions u, v possess partial derivatives at $(x, y) = (0, 0)$, and these satisfy $v_x + u_y = 0$, $v_y - u_x = 0$. But this means that (9 bis) holds at the point $(x, y) = (0, 0)$, since u and v vanish there.

Proof of Theorem ($\square$ bis). It follows from the preceding two proofs that (26) satisfies (27). Since

$$\iint_{|z|<R} dxdy/|z(z-\zeta)| = O(|\log|\zeta||) \text{ as } \zeta \to 0. \tag{28}$$

it follows that if $0 < R' < R$, there exists a constant $K = K(R')$ satisfying

$$|w(\zeta_1) - w(\zeta_2)| \leqq K|\zeta_2 - \zeta_1||\log|\zeta_2 - \zeta_1||$$

if $|\zeta_1| \leqq R'$, $|\zeta_2| \leqq R'$.

In particular, u and v satisfy on $|z| \leqq R'$ a uniform Hölder condition of every order $\mu < 1$. Since $\alpha, \beta, \gamma, \delta$ satisfy a uniform Hölder condition of order λ, it follows that $\alpha^* u + \beta^* v$ satisfies a uniform Hölder condition of order μ. The formula (27) is similar to that for the first derivatives of logarithmic potentials, with $\alpha^* u + \beta^* v$ having a rôle analogous to that of the density. Hence, w has partial derivatives satisfying a uniform Hölder condition of every order $\mu < \lambda$ on $|z| \leqq R'$. This proves Theorem ($\square$ bis).

8. The general linear case. The results of Section **2** can be trans-

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cribed from (2) to the case of the general elliptic, homogeneous, linear differential equation,

$$ar + 2bs + ct + dp + eq + fu = 0, \tag{29}$$

where

$$ac - b^2 > 0 \qquad (\text{and } a > 0), \tag{30}$$

provided that the functions d, e, f of (x, y) are, as above, just continuous, but the functions a, b, c are of class C^1 on (1) (as to the parenthetical normalization in (30), note that (29) can be multiplied by -1). Because of the C^1-assumption, (29) can be written in the form

$$(ap + bq)_x + (bp + cq)_y + dp + eq + fu = 0, \tag{31}$$

where d, e, f are continuous functions (not identical with the corresponding functions in (29)). In order to consider "solutions u of class C^1 of (31)," it is convenient to write (31) as an integral identity,

$$\int_J (bp + cq)\,dx - (ap + bq)\,dy = \iint_E (dp + eq + fu)\,dx\,dy, \tag{32}$$

where E is an arbitrary domain bounded by a piecewise smooth (C^1) Jordan curve J contained in (1).

Since a, b, c are of class C^1, it follows from the Lemma of [5], p. 761, that (32) is equivalent to

$$\int_J \Delta^{-1}\{(bp + cq)\,dx - (ap + bq)\,dy\} = \iint_E (dp + eq + fu)\,dx\,dy, \tag{33}$$

where

$$\Delta = (ac - b^2)^{\frac{1}{2}} > 0, \tag{34}$$

and d, e, f represent another set of continuous functions. Consider the system of partial differential equations

$$\theta_x = (b\phi_x + c\phi_y)/\Delta, \qquad \theta_y = -(a\phi_x + b\phi_y)/\Delta \tag{35}$$

for the unknown functions θ, ϕ. According to Lichtenstein [11], the fact that a, b, c are of class C^1 (hence satisfy a uniform Hölder condition on every compact subset of (1)) implies that the system (35) has a solution

$$\theta = \theta(x, y), \qquad \phi = \phi(x, y) \tag{36}$$

of class C^1 on (1) such that (36) is a topological mapping of (1) onto a (θ, ϕ)-domain D^* and has a non-vanishing Jacobian $j = \partial(\theta, \phi)/\partial(x, y)$. Actually, $j > 0$, since

$$j = a\theta_x^2 + 2b\theta_x\theta_y + c\theta_y^2 = a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 > 0. \tag{37}$$

In terms of a C^1-solution $u = u(x, y)$ of (33) and the transformation $x = x(\theta, \phi)$, $y = y(\theta, \phi)$ inverse to (36), define a function $U(\theta, \phi)$ on D^* by

$$U(\theta, \phi) \equiv u(x, y). \tag{38}$$

Then U is of class C^1 and, according to (33) and (34), satisfies

$$\int_J U_\phi d\theta - U_\theta d\phi = \iint_E (d^* U_\theta + e^* U_\phi + f^* U) d\theta d\phi, \tag{39}$$

where $d^* = (d\theta_x + e\theta_y)j$, $e^* = (d\phi_x + e\phi_y)j$, $f^* = fj$ are continuous functions of (θ, ϕ) and E is an arbitrary domain bounded by a piecewise smooth (C^1) Jordan curve J contained in D^*. The equation (39) is of the type (3).

REMARK. Note that even if $u(x, y)$ is a solution of class C^2 of (29), the function $U(\theta, \phi)$ need not be of class C^2. In fact, the transformation (36) need not be of class C^2 when it is only assumed that a, b, c are of class C^1; cf. [7], p. 265. Hence, even when starting with differential equations, (2) or (29), one is forced to consider the corresponding integro-differential equations, (3) or (32).

Any pair of functions $\theta^*(x, y)$, $\phi^*(x, y)$ for which $\theta^* + i\phi^*$ is a regular analytic function of $\theta + i\phi$ is a solution of (35). Hence, it can be supposed that θ and ϕ satisfy assigned "initial conditions" (consistent with (35)) at $(x, y) = (0, 0)$; for example,

$$\theta = \phi = 0 \text{ at } (x, y) = (0, 0) \tag{40}$$

and

$$\theta_x = \Delta,\ \theta_y = 0 \text{ and } \phi_x = -b,\ \theta_y = a \text{ at } (x, y) = (0, 0). \tag{41}$$

In this case,

$$\theta + i\phi \sim \Delta^0 x + i(a^0 y - b^0 x) \tag{42}$$

as $(x, y) \to (0, 0)$, where $a^0 = a(0, 0)$, $b^0 = b(0, 0), \cdots$ and

$$(U_\phi + iU_\theta)\Delta/a \sim \Delta^0 u_y + i(a^0 u_y - b^0 u_x). \tag{43}$$

Thus Theorems 1 and 2 can be transcribed as follows:

THEOREM 1*. *Let $a(x, y)$, $b(x, y)$, $c(x, y)$ be functions of class C^1 on* (1), *satisfying* (30), *hence, without loss of generality, the normalizations*

$$a(0, 0) = c(0, 0) = 1, \qquad b(0, 0) = 0, \tag{44}$$

and let $d(x, y)$, $e(x, y)$, $f(x, y)$ be continuous functions on (1). *Let $u = u(x, y)$ be a solution of class C^2 of* (29) *or, more generally, a function*

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of class C^1 satisfying (32) *for every domain E bounded by a piecewise smooth (C^1) Jordan curve J contained in* (1). *Then, if u satisfies* (4) *for some integer $n \geqq 0$, it must satisfy* (5) *(and therefore* (5′), (5″) *as well).*

The normalization (44) can always be accomplished by a linear transformation of the (x, y)-plane.

REMARK. We were unable to decide whether or not the assertions of Theorem 1*, and of Theorem 2* below, remain true if the coefficients of (29) are just continuous.

THEOREM 2*. *In addition to the conditions of Theorem* 1*, *assume that* $u \not\equiv 0$. *Then the conclusions of Theorem* 2 *are valid.*

Since $\operatorname{grad} U = (U_\theta, U_\phi)$ vanishes if and only if $\operatorname{grad} u = (u_x, u_y)$ vanishes at the corresponding point, the analogue of Corollary 1 holds.

COROLLARY 1*. *Under the assumptions of Theorems* 1* *and* 2*, *the conclusion of Corollary* 1 *is valid.*

9. The general elliptic case of a linear system. The application of a conformal mapping makes possible not only the generalization of Theorems 1 and 2, dealing with the equation (2), to Theorems 1* and 2*, dealing with the equation (29), but also a generalization of Theorem (*) to a theorem dealing with a general linear elliptic system:

THEOREM (**). *Let a, b, c, d be functions of class C^1 on* (1) *with the property that the matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{45}$$

has non-real roots and, without loss of generality, is normalized by the conditions

$$a(0,0) = d(0,0) = 0, \qquad b(0,0) = -c(0,0) = 1, \tag{45 bis}$$

and let α, β, γ, δ be continuous functions on (1). *Let u, v be continuous functions on* (1) *satisfying*

$$\int_J (au + by)\,dx + u\,dy = \iint_E (\alpha u + \beta v)\,dx\,dy, \tag{46$_1$}$$

$$\int_J (cu + dv)\,dx + v\,dy = \iint_E (\gamma u + \delta v)\,dx\,dy \tag{46$_2$}$$

for every domain E bounded by a piecewise smooth (C^1) Jordan curve J contained in (1) *(which means that*

$$(46\text{ bis}) \qquad u_x - (au + bv)_y = \alpha u + \beta v, \qquad v_x - (cu + dv)_y = \gamma u + \delta v,$$

if the six functions $a, \cdots, v$ are supposed to be of class C^1, for instance). Then, if (10_n) holds for some integer $n \geqq 0$, the conclusions of Theorem (*) are valid.

Proof of Theorem (**). Introduce the matrix and vector notation

$$(47) \qquad B = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then (46_1)-(46_2) can be written as a vector equation,

$$(48) \qquad \int_J A\sigma dx + \sigma dy = \iint_E B\sigma dx dy.$$

The assumptions on the matrix (45) show that if $\lambda \pm i\mu \equiv \lambda(x, y) \pm i\mu(x, y)$ are the characteristic numbers of (45), then there exists a non-singular matrix $T = T(x, y)$, of class C^1 on (1), satisfying

$$(49) \qquad TAT^{-1} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}, \qquad \mu \not\equiv 0,$$

and $\lambda(0, 0) = 0$, $\mu(0, 0) = 1$; in particular, λ and μ are of class C^1 on (1). In place of the dependent variable (vector) σ, introduce the new dependent variable

$$(50) \qquad \tau = \begin{pmatrix} U \\ V \end{pmatrix} = T\sigma.$$

Then (48) becomes

$$(51) \qquad \int_J AT^{-1}\tau dx + T^{-1}\tau dy = \iint_E BT^{-1}\tau dx dy.$$

Since T is of class C^1, the Lemma of [5], p. 761, shows that

$$(52) \qquad \int_J TAT^{-1}\tau dx + \tau dy = \iint_E \{TBT^{-1} + T_x AT^{-1} - T_y AT^{-1}\}\tau dx dy.$$

The first and second components of the vector on the left of (52) are the line integrals of $(\lambda U - \mu V)dx + Udy$ and $(\mu U + \lambda V) + Vdy$, respectively. Hence, if the second component is multiplied by i and added to the first, there results the scalar equation

$$(53)\qquad \int_J (U+iV)\{(\lambda+i\mu)dx+dy\} = \iint_E (\alpha^*U+\beta^*V)\,dxdy,$$

where α^*, β^* are (complex-valued) continuous functions of (x, y) on (1).

The system

$$(54)\qquad \theta_x = \lambda\theta_y - \mu\phi_y, \qquad \phi_x = \mu\theta_y + \lambda\phi_y$$

of partial differential equations in the unknown functions θ, ϕ can be written in the form

$$\theta_x = \mu^{-1}(\lambda\phi_x - (\lambda^2+\mu^2)\phi_y), \qquad \theta_y = \mu^{-1}(\phi_x - \lambda\phi_y).$$

The latter is of the Cauchy-Riemann-Beltrami type (35). Hence the theorem of Lichtenstein [11] implies the existence of a C^1-solution $\theta = \theta(x, y)$, $\phi = \phi(x, y)$ of (54) having the properties mentioned after (35).

Introduce θ, ϕ as new independent variables. Then, since (54) implies that $(\lambda+i\mu)dx+dy = (\theta_y+i\phi_y)^{-1}(d\theta+id\phi)$, the relation (53) becomes

$$\int_J (U+iV)(\theta_y+i\phi_y)^{-1}(d\theta+id\phi) = \iint_E (\gamma^*U+\delta^*V)\,d\theta d\phi,$$

where γ^*, δ^* are continuous functions of (θ, ϕ). If new dependent variables U^*, V^* are defined by $U^*+iV^* = (U+iV)(\theta_y+i\phi_y)^{-1}$, there results the identity

$$\int_J (U^*+iV^*)(d\theta+id\phi) = \iint_E (\alpha^{**}U^*+\beta^{**}V^*)\,d\theta d\phi,$$

where α^{**}, β^{**} are continuous functions of (θ, ϕ).

This shows that Theorem (*) is applicable if u, v, x, y in Theorem (*) are replaced by U^*, V^*, θ, ϕ, respectively. If the conclusions of Theorem (*) are re-stated in terms of the original variables u, v, x, y, Theorem (**) follows.

10. The Hessian. By generalizing an idea of Cohn-Vossen ([3]; cf. the proofs given by H. Hopf and H. Samelson [9] and by Zhitomirsky [15]) in the problem of embedding of a binary Riemannian analytic metric of positive curvature, H. Lewy [10] has formulated and proved a general theorem on the solutions of non-linear differential equation $F = 0$ of elliptic type, under the assumption that the function $F = F(x, \cdots, t)$ is analytic in its eight variables and the solutions are analytic in (x, y). *Loc. cit.*, the restriction of analyticity is essential indeed; the proof ([10], pp. 259-260) does not apply even if the solutions are of class C^∞. It is therefore of interest that the procedures applied above lead to a proof which extends Lewy's theorem to the

case in which F is of class C^2 only and the solutions involved are of class C^3 (Theorem (†) in Section **12** below).

To this end, the following counterpart of Theorem 1 will first be proved:

THEOREM 3*. *Besides the assumptions made in Theorem* 1* *on the coefficient functions* $a, \cdots, f$ *of* (29), *suppose that* d, e, f, *too, are of class* C^1 *on* (1). *Then, if* u *is any solution, of class* C^2, *of* (29) *satisfying* (4) *for some positive integer* n, *both limits*

$$(55)\qquad \lim_{\rho\to 0}(u_{xy}+iu_{xx})/(x+iy)^{n-1}, \qquad \lim_{\rho\to 0}(u_{yy}+iu_{xy})/(x+iy)^{n-1} \text{ exist}$$

(*for the same* $n \geqq 1$).

If the two limits (55) are denoted by $n(c_2+ic_1)$, $n(c_4+ic_3)$, where c_k is real, then, since $\rho=(x^2+y^2)^{\frac{1}{2}}$, it follows from (55) that, to an error $o(\rho^{1-n})$ as $\rho\to 0$, the function $\rho^{1-n}u_{xy}/n$ of (x,y) is both

$$c_2\cos(n-1)\theta - c_1\sin(n-1)\theta \text{ and } c_3\cos(n-1)\theta + c_4\sin(n-1)\theta,$$

where $x+iy=\rho e^{i\theta}$. Hence $c_2-c_3=0$ and $c_1+c_4=0$. Consequently, from (55),

$$\begin{aligned} u_{xx} &= n\rho^{n-1}(c_1\sin(n-1)\theta + c_2\sin(n-1)\theta) + o(\rho^{n-1}),\\ u_{xy} &= n\rho^{n-1}(c_2\cos(n-1)\theta - c_1\sin(n-1)\theta) + o(\rho^{n-1}),\\ u_{yy} &= n\rho^{n-1}(-c_1\cos(n-1)\theta - c_2\sin(n-1)\theta) + o(\rho^{n-1}). \end{aligned} \qquad (55\text{ bis})$$

Clearly, (5′) follows from (55 bis) in the same way as (5″) did from (5′).

Note that the assumption in Theorem 3* is $n \geqq 1$ (whereas it is $n \geqq 0$ in Theorem 1*); so that grad $u=0$ at $(x,y)=(0,0)$. This assumption cannot, in general, be replaced by

$$(56)\qquad u(x,y) = u(0,0) + p(0,0)x + q(0,0)y + o(\rho^n);$$

cf. the counterexamples given in connection with (8), Section **2**. Of course, it need not be assumed in Theorem 3* that $u(0,0)=0$ if $f\equiv 0$; similarly, (56) can replace (4) if $d\equiv e\equiv f\equiv 0$.

Since Theorem 2* shows that the limits in (55) cannot be 0 for *every* positive integer n when $u\not\equiv 0$, it follows that the "flat points" of the surface $u=u(x,y)$, that is, the points where the Hessian matrix of u is the zero matrix, cannot cluster at $(x,y)=(0,0)$ unless the surface is the plane $u\equiv 0$. In this direction, Theorem 3* contains a stronger statement:

COROLLARY 2*. *Under the assumptions of Theorem 3*, there exists an* $\epsilon > 0$ *such that*

$$u_{xx}u_{yy} - u_{xy}^2 < 0 \text{ for } 0 < x^2 + y^2 < \epsilon^2 \tag{57}$$

unless $u \equiv 0$.

In other words, the Gaussian curvature of the surface $u = u(x, y) \not\equiv 0$ is negative in a punctured vicinity of $(x, y) = (0, 0)$. Thus $(x, y) = (0, 0)$ is either a hyperbolic or an *isolated* parabolic point of the surface.

11. Proof of Theorem 3. It can be assumed that

$$\Delta = (ac - b^2)^{\frac{1}{2}} \equiv 1, \tag{58}$$

since (29) can be multiplied by $1/\Delta$.

If u is of class C^3, differentiation of (29) with respect to x, y and an application of Green's formula give the identities

$$\int_J (br + cs)\,dx - (ar + bs)\,dy = \iint_E I_1\,dxdy, \tag{59}$$

$$\int_J (bs + ct)\,dx - (as + bt)\,dy = \iint_E I_2\,dxdy, \tag{60}$$

where E is any subdomain E of (1) bounded by a piecewise smooth (C^1) Jordan curve J in (1), and

$$I_1 = (d - b_y)r + (e + b_x - c_y)s + c_x t + d_x p + e_x q + f_x u, \tag{61}$$

$$I_2 = a_y r + (d + b_y - a_x)s + (e - b_x)t + d_y p + e_y q + f_y u. \tag{62}$$

Actually, these integral relations can be proved under the assumptions of Theorem 3*, where it is only supposed that u is of class C^2.

To this end, note that (29) gives

$$-\int_J (ar + bs)dy = \int_J bs dy + \int_J ct dy + \int_J (dp + eq + fu)dy. \tag{63}$$

The last integral can be written as a double integral,

$$\int_J (dp + eq + fu)dy = \iint_E (dr + es + fp + d_x p + e_x q + f_x u)dxdy. \tag{64}$$

The first integral on the right of (63) can be treated as follows:

$$\int_J bsdy = -\int_J brdx + \int_J b(rdx + sdy),$$

where

$$\int_J b(rdx + sdy) = \iint_E (b_x s - b_y r)\,dxdy,$$

by the Lemma in [5], p. 761. Thus

$$\int_J bsdy = -\int_J brdx + \iint_E (b_x s - b_y r)\,dxdy. \tag{65}$$

Similarly,

$$\int_J ctdy = -\int_J csdx + \iint_E (c_x t - c_y s)\,dxdy. \tag{66}$$

But (59) follows from (63), (64), (65) and (66). The relation (60) is proved similarly.

Let (36) be a solution of (35) such that (36) has a non-vanishing Jacobian, maps (1) onto some domain D^* in a one-to-one manner and (40) holds. Introduce the abbreviations

$$U(\theta,\phi) \equiv u(x, y), \qquad P(\theta, \phi) \equiv p(x, y), \qquad Q(\theta, \phi) \equiv q(x, y). \tag{67}$$

Then (59), (60) can be writen as

$$\int_J P_\phi d\theta - P_\theta d\phi = \iint_E I^*_1 d\theta d\phi, \tag{68}$$

$$\int_J Q_\phi d\theta - Q_\theta d\phi = \iint_E I^*_2 d\theta d\phi, \tag{69}$$

where I^*_1, I^*_2 are linear forms in U, P, Q, P_θ, P_ϕ, Q_θ, Q_ϕ, with coefficients which are continuous functions of (θ, ϕ). Also

$$U(\theta, \phi) = \int_{(0,0)}^{(\theta,\phi)} (Px_\theta + Qy_\theta)\,d\theta + (Px_\phi + Qy_\phi)\,d\phi.$$

For a moment, let (θ, ϕ) be renamed (x, y). Then (14) holds for $w = w_1$ or $w = w_2$ and $W = W_1$ or $W = W_2$, respectively, where $w_1 = P_y + iP_x$, $W_1 = I^*_1$ and $w_2 = Q_y + iQ_x$, $W_2 = I^*_2$. Correspondingly, (17_k) holds for

$w = w_1$ or $w = w_2$ if (16_k) holds for $w = w_1$ or $w = w_2$, respectively. If $R > 0$ is sufficiently small, then there exists a constant $K = K_R > 0$ satisfying

$$|W_1| + |W_2| \leqq K(|U| + |P| + |Q| + |w_1| + |w_2|) \text{ for } |z| \leqq R.$$

Hence, if $|w_1| + |w_2|$ is denoted by ω, the inequality corresponding to (19_k) is

$$2\pi\omega(\zeta)|\zeta|^{-k} \leqq \int_{|z|=R} \omega\,|z^{-k}(z-\zeta)^{-1}|\cdot|dz| + K\iint_{|z|<R} (\omega + |P| + |Q| + |U|)\,|z^{-k}(z-\zeta)^{-1}|\,dxdy.$$

Since $|U| + |P| + |Q| = o(\rho^{1-n})$ as $\rho \to 0$, obvious modifications of the last part of the proof of Theorem 1 show that w_1/z^{n-1} and w_2/z^{n-1}, where $z = x + iy$, tend to limits as $z \to 0$.

In the original notation, this means that, as $(\theta, \phi) \to (0, 0)$,

$$\lim(P_\phi + iP_\theta)/(\theta + i\phi)^{n-1} \text{ and } \lim(Q_\phi + iQ_\theta)/(\theta + i\phi)^{n-1}$$

exist. It is seen from the proof of Theorem 2* that this completes the proof of (55), which is the statement of Theorem 3*.

Proof of Corollary 2*. Let $n(>1)$ be the least positive integer for which not both limits (55) are 0; that is, the least positive integer for which (4) does not hold. Then not both constants c_1, c_2 are 0 in (55 bis) and (5′), (5″). Since (55 bis) implies that

$$u_{xx}u_{yy} - u_{xy}^2 = -\rho^{2n-2}(c_1^2 + c_2^2) + o(\rho^{2n-2}),$$

Corollary 2* follows.

12. Non-linear elliptic equations. It has been observed by Hadamard [4], pp. 352-354, that, in proving uniqueness theorems for partial differential equations

$$F(x, y, u, p, q, r, s, t) = 0, \tag{70}$$

it is usually sufficient to consider the linear cases (29). For, if F in (70) is a function of class C^n, with $n \geqq 1$, on some eight-dimensional, convex domain $\mathfrak{D}$, and if $u = u_1(x, y)$ and $u = u_2(x, y)$ are solutions of (70) of class C^k, where $k \geqq 2$, on some (x, y)-domain D, then the difference $u = u_1 - u_2$ is a solution of a linear equation (29), in which the functions

$a, b, \cdots, f$ of (x, y) are of class C^m, where $m = \min(n-1, k-2)$. If in addition $4F_rF_t - F_s^2 \neq 0$ on $\mathcal{D}$, then $\operatorname{sgn}(ac - b^2) = \operatorname{sgn}(4F_rF_t - F_s^2)$. Hence, Theorems 1*, 2*, 3* and their Corollaries imply the following theorem:

THEOREM (†). *Let $F(x, y, u, p, q, r, s, t)$ be a function of class C^2 on an eight-dimensional convex domain $\mathcal{D}$ on which*

$$4F_rF_t - F_s^2 > 0. \tag{71}$$

On some connected (x, y)-domain D, let $u_1(x, y)$ and $u_2(x, y)$ be two solutions of (70), both of class C^3, and suppose that the function $u = u_1 - u_2$ of (x, y) satisfies

$$u = 0 \text{ at } (x, y) = (0, 0) \tag{72}$$

but $u \not\equiv 0$ (i. e., $u_1 \not\equiv u_2$) on D. Then (4) cannot hold for every n and, if $\epsilon > 0$ is small enough,

$$u_x^2 + u_y^2 \neq 0 \text{ for } 0 < x^2 + y^2 < \epsilon^2. \tag{73}$$

If, in addition to (72), it is assumed that

$$u_x^2 + u_y^2 = 0 \text{ at } (x, y) = (0, 0), \tag{74}$$

then ϵ can be chosen so small that, in addition to (73),

$$u_{xx}u_{yy} - u_{xy}^2 < 0 \text{ for } 0 < x^2 + y^2 < \epsilon^2. \tag{75}$$

In his short note [1], Carleman states that a weakened form of the conclusion (73), namely, $u^2 + u_x^2 + u_y^2 \neq 0$ for $0 < x^2 + y^2 < \epsilon^2$, follows from his result on linear elliptic systems of the form (9 bis), where $\alpha, \beta, \gamma, \delta$ are continuous functions of (x, y). But this argument does not seem to be obvious. It is certainly valid if the linear partial differential equation (29) satisfied by the difference $u = u_1 - u_2$ is of the type (2), with $f \equiv 0$. For then (2) can be written as the system $q_x - p_y = 0$, $q_y + p_x = -eq - dp$. But if the coefficients a, b, c are only of class C^1, then (29) cannot in general be reduced to the form (2), but only to the integrated form (3) of (2); cf. the Remark following (39). After this difficulty, the case $f \not\equiv 0$ would still remain to be dealt with.

In this connection it is worth mentioning that *the assumption (72) is not needed for the conclusions (73) and (75) if F does not depend on u explicitly* and that *neither of the assumptions (72), (74), is needed for (75) if F does not contain (u, p, q).*

If $u = u_1(x, y)$ and $u = u_2(x, y)$ are analytic functions (i. e., expand-

able in power series) of (x, y) and if F, too, is an analytic function (of $(x, y, u, \cdots, t)$), then the conclusion (75) is due to Lewy [10], pp. 259-260.

13. Non-parabolic linear systems. In what follows, a uniqueness theorem on linear systems will be considered. It will be proved under assumptions of differentiability lighter than those imposed by Carleman [2]. This will depend on a suitable modification and, at the same time, simplification of Carleman's proof (some points omitted by Carleman [2], p. 8, will be proved in detail; cf. the motivations in (87)-(90) below). The refined form of Carleman's theorem is as follows:

THEOREM (§). *Let $A = A(x, y)$ and $B = B(x, y)$ be n by n matrix functions of class C^1 and of class C^0 (= continuous), respectively, on the closure of the (x, y)-domain*

$$D_R\colon\ x > 0,\ x^2 + y^2 < R^2, \tag{76}$$

and let the elementary divisors of A be simple. Let $u = u(x, y)$ be any vector function, with n components, which is of class C^1 and satisfies

$$u_x + Au_y = Bu \tag{77}$$

on (76), is continuous on the closure of (76) and, if $u(0, y)$ denotes the boundary value $u(+\,0, y)$, let

$$u(0, y) \equiv 0 \tag{78}$$

for $|\,y\,| < R$. Then there exists a positive $\epsilon (< R)$ such that $u(x, y) \equiv 0$ on D_ϵ.

The proof of Theorem (§) will use the fact that, by the assumptions made for A, the definition of the function A can be extended from the domain D_R in such a way that A becomes of class C^1 on a domain containing the closure of D_R. In the statement of his theorem, Carleman ([2], p. 1) requires that A be of class C^2 (instead of class C^1) and that B be of class C^0 on D_R (not on the closure of D_R). Carleman makes no assumption on A and B as $x \to +\,0$, but this must be an oversight; for he obviously uses the fact that B is bounded on (76) and, in his use of Green's formula, he implicitly imposes (unspecified) conditions on A as $x \to +\,0$.

REMARK. Corresponding to the Remark following Corollary 2 in Section 1, Theorem (§) remains true if the linear terms Bu on the right of (77) are modified by continuous non-linear terms satisfying conditions similar to those specified by the formula line in that Remark. This will be clear from the proof of Theorem (§).

14. Proof of Theorem (§). The assumptions made for A imply that there exists on the closure of D_R a non-singular matrix $T = T(x, y)$, of class C^1, for which $T^{-1}AT$ becomes a matrix with binary blocks of the form

$$\begin{pmatrix} \lambda_j & -\mu_j \\ \mu_j & \lambda_j \end{pmatrix}, \quad \mu_j \neq 0, \qquad j = 1, \cdots, m \tag{79}$$

along its main diagonal, where $0 \leqq m \leqq \frac{1}{2}n$, and with $n - 2m$ diagonal element λ_j, where $j = 2m + 1, \cdots, n$, while all other elements of $T^{-1}AT$ are zero; cf. [6], p. 856. Hence, if u is replaced by the vector Tu, it can be supposed that A itself has the normal form, just described. Thus, if $u = (u^1, \cdots, u^n)$ and $B = (b_{jk})$, the system (1) is of the form

$$u^j{}_x + \lambda_j u^j{}_y - \mu_j u^{j+1}{}_y = \sum_{k=1}^{n} b_{jk} u^k, \qquad \text{where } j = 1, 3, \cdots, 2m - 1 \tag{80$_j$}$$

$$u^{j+1}{}_x + \mu_j u^j{}_y + \lambda_j u^{j+1}{}_y = \sum_{k=1}^{n} b_{j+1\,k} u^k$$

and

$$u^j{}_x + \lambda_j u^j{}_y = \sum_{k=1}^{n} b_{jk} u^k, \qquad \text{where } j = 2m + 1, \cdots, n. \tag{81$_j$}$$

The functions $\mu_1, \cdots, \mu_m$; $\lambda_1, \cdots, \lambda_m$; $\lambda_{2m+1}, \cdots, \lambda_n$ are of class C^1 on the closure of D_R and can, therefore, be extended so as to become of class C^1 on (1).

It can be supposed that $m > 0$, for otherwise (77) is a hyperbolic system for which Theorem (§) seems to be well-known, even though B is just continuous (cf. a statement of Carleman [2], p. 2; the assumptions made by Holmgren [8] are heavier; see also [6], pp. 855-864, where on p. 864, the first reference to Holmgren is erroneous and must be replaced by [8] in the bibliography of the present paper). There corresponds to the pair $(\lambda, \mu) = (\lambda_j, \mu_j)$ the Cauchy-Riemann-Beltrami system (54), which has on (1) a solution $\theta = \theta_j(x, y)$, $\phi = \phi_j(x, y)$, of class C^1 and of non-vanishing Jacobian, satisfying

$$\theta_j(0, 0) = \phi_j(0, 0) = 0 \tag{82}$$

and mapping (1) in a one-to-one manner on a (θ_j, ϕ_j)-domain D^j.

Let $c > 0$, $\tau > 1$ be such that the portion $x > 0$ of the circle $| z + c | \leqq \tau c$, where $z = x + iy$, is contained in D_R. Let this portion of D_R be denoted by $D(c)$, and let $\Gamma(c)$ denote the arc $x > 0$ of the circle $| z + c | = \tau c$; finally, let the images of $D(c)$, $\Gamma(c)$ under the transformation $(x, y) \to (\theta_j, \phi_j)$ be

called $D^j(c)$, $\Gamma^j(c)$, respectively. The pair of equations (80_j) on $D(c)$ is transformed into equations of the form

$$u^j{}_\theta - u^{j+1}{}_\phi = \sum_{k=1}^{n} b_k u^k, \qquad u^j{}_\phi + u^{j+1}{}_\theta = \sum_{k=1}^{n} b^*{}_k u^k, \qquad (\theta, \phi) = (\theta_j, \phi_j),$$

on $D^j(c)$, where b_k, $b^*{}_k$ and u are continuous functions of (θ, ϕ) on the closure of $D^j(c)$. (Incidentally, the coefficients b_k, $b^*{}_k$ depend in a very simple manner on b_{jk}, $b_{j+1\,k}$, on the partial derivatives θ_x, θ_y, ϕ_x, ϕ_y and on the Jacobian $\partial(\theta, \phi)/\partial(x, y) \neq 0$.) It follows that there exists a constant K (independent of j and c, for small $c > 0$) for which

$$\iint_D |\, w^j(\zeta)/(\zeta - \zeta_0)^l \,|\, d\theta d\phi \tag{83}$$

$$\leqq Kc \int_\Gamma |\, w^j(\zeta)/(\zeta - \zeta_0)^l \,| \cdot |\, d\zeta \,| + Kc \iint_D \sum_{k=1}^{n} |\, u^k/(\zeta - \zeta_0)^l \,|\, d\theta d\phi,$$

where l is a positive integer, $w^j = u^{j+1} + iu^j$, $D = D^j(c)$, $\Gamma = \Gamma^j(c)$, $\zeta = \theta + i\phi = \theta_j + i\phi_j$, and ζ_0 is the $(\theta_j + i\phi_j)$-image of $x + iy = -c + i0$; cf. the derivation of (24) above.

Change the integration variables from $(\theta, \phi) \equiv (\theta_j, \phi_j)$ to (x, y) in (83). The function $|\, \partial(\theta, \phi)/\partial(x, y) \,|$ is bounded from above and from below by positive constants (independent of c, for small $c > 0$, on $D(c)$); also, the ratio $|\, \zeta - \zeta_0 \,|/|\, z + c \,|$ is bounded from above and from below by positive constants; finally, $|\, d\zeta \,|$ is majorized by Const. $|\, dz \,|$, since $(d\theta + id\phi) = (\theta_y + i\phi_y)\, dz$. Hence, (83) implies that, if the constant K is large enough,

$$\iint_D \{(|\, u^j \,| + |\, u^{j+1} \,|)/|\, z + c \,|^l\} dxdy \tag{84_j}$$

$$\leqq Kc \int_\Gamma \{(|\, u^j \,| + |\, u^{j+1} \,|)/|\, z + c \,|^l\} |\, dz \,| + Kc \sum_{k=1}^{n} \iint_D |\, u^k/(z + c)^l \,|\, dxdy,$$

where $D = D(c)$ and $\Gamma = \Gamma(c)$. Addition of $(84_1), (84_3), \cdots, (84_{2m-1})$ gives

$$(1 - Kc) \sum_{j=1}^{2m} \iint_D |\, u^j/(z + c)^l \,|\, dxdy \tag{85}$$

$$\leqq Kc \sum_{j=1}^{2m} \int_\Gamma |\, u^j/(z + c)^l \,| \cdot |\, dz \,| + Kc \sum_{k=2m+1}^{n} \iint_D |\, u^k/(z + c)^l \,|\, dxdy,$$

where $D = D(c)$ and $\Gamma = \Gamma(c)$.

In order to appraise the last integral, let $y = y_j = y(x;\xi,\eta)$, where $y(\xi;\xi,\eta) = \eta$, denote the (unique) solution of the differential equation

$$(86_j) \qquad dy/dx = \lambda_j(x,y) \qquad (j = 2m+1, \cdots, n)$$

of the "characteristics" of (81_j). Then, according to (81_j) and the assumption $u^j(0,y) \equiv 0$,

$$(87) \qquad u^j(x,y) = \sum_{k=1}^{n} \int_0^x b_{jk}(t, y(t;x,y))u^k(t, y(t;x,y))dt.$$

If c and τ — (>0) are small, the absolute value of the slope of the arc $\Gamma(c)$ exceeds any given constant, hence, in particular, the function $|\lambda_j(x,y)|$. Hence (86_j) implies that if the point (x,y) is in $D(c)$, then the arc $(t, y(t;x,y))$, where $0 < t < x$, is in $D(c)$; in fact, $|t + iy(t;x,y) + c|$ is an increasing function of t. Consequently, the constant K can be chosen so large that

$$(88) \qquad |u^j(x,y)/(z+c)^l|$$
$$\leqq K \sum_{k=1}^{n} \int_0^x |u^k(t, y(t;x,y))/(t + c + iy(t;x,y))^l| \, dt$$

for all (x,y) in $D(c)$. Integration of this inequality with respect to $dxdy$ over $D = D(c)$ gives

$$(89) \qquad \iint_D |u^j/(z+c)^l| \, dxdy$$
$$\leqq K \sum_{k=1}^{n} \iint_D \int_0^x |u^k(t; y(t;x,y))/(t + c + iy(t;x,y))^l| \, dtdxdy.$$

Let the integration variables (x,y) in the triple integral on the right of (89) be replaced by the variables (ρ,θ), where $z + c = \rho e^{i\theta}$. Then, if $\theta = \theta(\rho) = \arccos c/\rho$ and $0 < \rho(\theta) < \frac{1}{2}\pi$, the triple integral becomes of the type

$$\int_c^{\tau c} \rho d\rho \{ \int_{-\theta(\rho)}^{\theta(\rho)} \int_0^{\rho\cos\theta} h(t; y(t; \rho\cos\theta, \rho\sin\theta))dtd\theta \}$$

where $h = h(x,y) = |u^k(x,y)/(z+c)^l|$. If the integration variables (t,θ) in the inner double integral are replaced by (x,y), where $x = t$, $y = y(t; \rho\cos\theta, \rho\sin\theta)$, then the domain of integration for (x,y) is $E: x > 0$, $|z + c| < \rho$; cf. the remark following (87). Furthermore, the Jacobian

$$\partial(x, y)/\partial(t, \theta) = \rho \sin\theta\, \partial y(t; \xi, \eta)/\partial\xi + \rho\cos\theta\, \partial y(t; \xi, \eta)/\partial\eta,$$

where $(\xi, \eta) = (\rho\cos\theta, \rho\sin\theta)$, lies between two positive constant multiples of ρ, since $|\sin\theta|$ is small and $\partial y(0; 0, \eta)/\partial\eta \equiv 1$. Hence the last triple integral is majorized by

$$K\int_c^{\tau c} d\rho\{\iint_E h(x, y)dxdy\},$$

where K is a constant. Consequently, (89) leads to the inequality

$$\iint_D |\, u^j/(z+c)^l\,|\, dxdy \leqq Kc\sum_{k=1}^{n}\iint_D |\, u^k/(z+c)^l\,|\, dxdy,$$

where $D = D(c)$, $j = 2m+1, \cdots, n$, and $l = 1, 2, \cdots$.

Addition of these inequalities gives

$$(90)\qquad (1 - Kc)\sum_{j=2m+1}^{n}\iint_D |\, u^j/(z+c)^l\,|\, dxdy \leqq Kc\sum_{k=1}^{2m}\iint_D |\, u^k/(z+c)^l\,|\, dxdy,$$

where $D = D(c)$ and $l = 1, 2, \cdots$. If $c > 0$ is so small that $1 - Kc > \frac{1}{2}$, then the last inequality, when combined with (85), leads to

$$(91)\qquad (1 - 3Kc)\sum_{j=1}^{2m}\iint_D |\, u^j/(z+c)^l\,|\, dxdy \leqq Kc\sum_{k=1}^{2m}\int_\Gamma |\, u^k/(z+c)^l\,|\cdot|\, dz\,|.$$

As at the end of the proof of Theorem 2, it follows that (91) cannot hold for large l unless $u^j \equiv 0$ on $D = D(c)$ for $j = 1, 2, \cdots, 2m$, in which case (90) shows that $u^j \equiv 0$ on $D = D(c)$ for $j = 2m+1, \cdots, n$. This proves Theorem (§).

REMARK. If (77) is an elliptic system, that is, if no eigenvalue of $A(x, y)$ is real, then the hypothesis that u vanishes on a segment of the y-axis can be replaced by the assumption that $u = 0$ holds on a sequence of point of (76) clustering at an interior point of the domain of definition of u. (Actually, if, in the wording of Theorem (§) above, u is defined to be iden-

tically 0 for $x < 0, x^2 + y^2 < R^2$, then u is a continuous solution of an integrated form of (77) on (1).) On the other hand, if A has at least one real characteristic number, then the assumption $u \equiv 0$ for $x = 0$ cannot be replaced by the assumption that $u = 0$ holds on a set of points clustering at a point of D_R. In order to see this, it is sufficient to choose (77) to be a scalar equation, say $u_x = 0$.

15. Non-parabolic non-linear systems. The linearizing device of Hadamard, used above (Section **12**) in the reduction of Theorem (†) to Theorems 1*, 2*, 3*, also shows that Theorem (§) has the following consequence:

THEOREM (∇). *Let $F = F(x, y, u, w)$, where F, u, w are vectors with n components, be a function of class C^2 on a $(2n + 2)$-dimensional domain $\mathfrak{D}$ which contains the product set of the (x, y)-domain D_R defined by* (76), *and of a $2n$-dimension convex (u, w)-domain, and suppose that the elementary divisors of the Jacobian matrix* $\operatorname{grad}_w F$ *are simple at every point of $\mathfrak{D}$. On D_R, let $u = u_1(x, y)$ and $u = u_2(x, y)$ be two solutions, of class C^2, of the system of n differential equations*

$$u_x = F(x, y, u, u_y). \tag{92}$$

Then, if $u_1(+0, y)$, $u_2(+0, y)$ exist uniformly and are identical for $-R < y < R$, there exists an $\epsilon > 0$ such that $u_1(x, y) \equiv u_2(x, y)$ on D_ϵ.

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ON PIECES OF CONVEX SURFACES.*

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1. Let the vector function $N = N(u, v)$ be either of the continuous unit normals of a sufficiently small piece S of a surface which is of class C^1 and is given in the parametric form $X = X(u, v)$, where $X = (x, y, z)$ is a function of class C^1 on a simply connected (u, v)-domain and the vector product (X_u, X_v) does not vanish. If $X = X(u, v)$ is of class C^2 and the Gaussian curvature $K = K(u, v)$ is not zero, then the correspondence between points $X(u, v)$ of S and points $N(u, v)$ of the unit sphere, $|N| = 1$, is one-to-one in a neighborhood of every point (u^0, v^0). Such a correspondence can exist when the assumption $K \neq 0$ is weakened to $K \geqq 0$ or $K \leqq 0$, but cannot exist if K assumes both positive and negative values in every neighborhood of the point (u^0, v^0) under consideration.

In order to verify this negative statement, note that if

$$p = p(x, y), \qquad q = q(x, y) \tag{1}$$

is a one-to-one continuous mapping of a simply connected (x, y)-domain D onto a (p, q)-domain, then the orientation of the image in the (p, q)-plane of a positively oriented Jordan curve J in D is independent of J. If in addition the functions (1) are of class C^1 and if (1) has a non-vanishing Jacobian, then the orientation of the image in the (p, q)-plane is positive or negative according as the Jacobian $\partial(p, q)/\partial(x, y)$ is positive or negative. Hence, if the functions (1) are of class C^1, then (1) cannot be a one-to-one, continuous mapping of one simply connected domain onto another when $\partial(p, q)/\partial(x, y)$ assumes positive and negative values.

If a surface S of class C^1 has a parametrization $X = (x, y, z(x, y))$, where $z(x, y)$ is of class C^1 on a (small) simply connected domain D, then the correspondence between points X of S and points N of the unit sphere is essentially given by (1), where

$$p = z_x(x, y), \qquad q = z_y(x, y). \tag{2}$$

In fact, the direction cosines (λ, μ, ν) of $\pm N$ are determined by

$$\lambda = p/(1 + p^2 + q^2)^{\frac{1}{2}}, \qquad \mu = q/(1 + p^2 + q^2)^{\frac{1}{2}} \tag{3}$$

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* Received September 25, 1952.

and $\lambda^2+\mu^2+\nu^2=1$; so that the mapping $X\to N$ is continuous and one-to-one if and only if (2) is. When S (hence z) is of class C^2, the Jacobian of (2) is $z_{xx}z_{yy}-z_{xy}{}^2$, which is $(1+p^2+q^2)^2$ times the Gaussian curvature $K=K(x,y)$. Since this implies that $\operatorname{sgn}\partial(p,q)/\partial(x,y)=\operatorname{sgn}K$, the negative statement, made before (1), follows.

2. There will be considered an analogue of the statement in the first paragraph concerning the correspondence $X\to N$ when $K\neq 0$. In this analogue, it will only be assumed that S is of class C^1; in particular, S need not possess a Gaussian curavture K. The assumption $K>0$ or $K<0$ will be replaced by the hypotheses that S is strictly convex (which implies that $K\geqq 0$ if S is of class C^2).

A continuous function

$$z=z(x,y), \tag{4}$$

defined on a convex neighborhood D of $(x,y)=(0,0)$, is called *strictly convex* if

$$z(\tfrac{1}{2}\{x_1+x_2\},\tfrac{1}{2}\{y_1+y_2\})<\tfrac{1}{2}\{z(x_1,y_1)+z(x_2,y_2)\} \tag{5}$$

holds for every pair of distinct points (x_1,y_1), (x_2,y_2) of D. The function (4) is called convex if the $<$ in (5) is weakened to $\leqq$. A surface S will be called [strictly] convex if, after a suitable rotation of the (x,y,z)-space, it has a parametrization $X=X(x,y,z(x,y))$ in which (4) is [strictly] convex.

Let S be a surface of class C^1. By a spherical C^0-parametrization $X=X(\lambda,\mu)$ of S is meant a continuous, one-to-one mapping of a plane (λ,μ)-domain onto the surface S in such a way that λ, μ are two of the three direction cosines of a unit normal N at the point $X(\lambda,\mu)$ of S; for example, $N=(\lambda,\mu,\nu)$; cf. [2], pp. 305-306.

(I) *Let S be a strictly convex surface of class C^1. Then S has a spherical C^0-parametrization $X=X(\lambda,\mu)$ and, with respect to this parametrization, the supporting function*

$$H=X\cdot N \tag{6}$$

is of class C^1 (even though $X(\lambda,\mu)$ is just of class C^0).

It can be supposed that S has a parametrization $X=(x,y,z(x,y))$ in which (4) is of class C^1 and strictly convex in a convex neighborhood D of $(x,y)=(0,0)$. The supporting function (6), in terms of (x,y), is given by

$$H=h/(1+p^2+q^2)^{\frac{1}{2}}, \tag{7}$$

where

$$h = xp + yq - z, \tag{8}$$

and p, q are defined by (2). In view of the remark made above on (2)-(3), the assertion (I) will be proved if it is shown that the transformation $(x, y) \to (p, q)$, defined by (2), is a continuous, one-to-one mapping of D onto a (p, q)-domain, and that h is of class C^1 as a function of (p, q). Hence (I) is contained in the following assertion:

(II) *Let $z = z(x, y)$ be a strictly convex function of class C^1 on a convex domain D. Then (2) is a continuous, one-to-one mapping of D onto a (p, q)-domain $\mathfrak{D}$, and (8), as a function of (p, q), is of class C^1 and strictly convex on every convex subdomain of $\mathfrak{D}$; finally, the inverse of (2) is*

$$x = h_p(p, q), \qquad y = h_q(p, q). \tag{9}$$

Thus the Legendre transformation $(x, y; z) \to (p, q; h)$, *along with the C^1-assumption*, is involutory.

The assertion of (II) has been proved by H. Lewy [3], pp. 365-366, under more restrictive conditions on z; namely, under the assumptions that z is of the form $z = \frac{1}{2}(x^2 + y^2) + z_1(x, y)$, where $z_1(x, y)$ is convex and of class C^1. The latter assumptions mean that, if z is smooth, the Gaussian curvature of the surface (4) satisfies $K > 0$, whereas the assumption of strict convexity only assures that $K \geqq 0$. It should also be noted that the involutory character, pointed out after (9), is lost if the functions z considered are those satisfying the conditions of Lewy's theorem (rather than the condition of strictly convex C^1-functions).

REMARK. *Theorems* (I), (II), *as well as* (IV) *below, have analogues for the case in which the surface S is replaced by a plane curve* Γ. Actually, these Γ-variants are corollaries of the S-theorems, since they result when S is a surface obtained by revolving Γ about a suitable axis.

3. The assertions of (I), (II) cannot be improved:

(III) *If the surface S: $z = z(x, y)$ is analytic (or even a polynomial) and strictly convex, then its supporting function H need not be of class C^2 in terms of spherical parameters; that is,* (8) *need not be of class C^2 as a function of (p, q).*

On the other hand, if S: $z = z(x, y)$ is of class C^n, where $n \geqq 2$, with a non-vanishing curvature K, then the supporting function is of class C^n as a function of (p, q); cf. [4], Appendix.

The truth of (III) is shown by the example $z(x, y) = (x^4 + y^4)/4$. In fact, an easy calculation shows that the corresponding function $h = h(p, q)$ is $3/4$ times $p^{4/3} + q^{4/3}$ and that the supporting function $H(p, q)$ is $3/4$ times $(p^{4/3} + q^{4/3})/(1 + p^2 + q^2)^{\frac{1}{2}}$.

It is worth observing that, in this example, (2) maps the convex domain D: $x^2 + y^2 < 1$ onto the non-convex domain $\mathfrak{D}$: $p^{2/3} + q^{2/3} < 1$.

4. *Proof of* (II). The assumptions made on $z(x, y)$ imply that if (x, y) is any point of D and θ is arbitrary, then $z(x + t\cos\theta, y + t\sin\theta)$ has a continuous derivative, $z_x\cos\theta + z_y\sin\theta$, with respect to t and that this derivative is strictly increasing with t. Hence, if (x', y'), (x'', y'') are two distinct points of D, then

$$z_x(x'', y'')\cos\theta + z_y(x'', y'')\sin\theta > z_x(x', y')\cos\theta + z_y(x', y')\sin\theta,$$

where $(\cos\theta, \sin\theta)$ denotes the unit vector in the direction of $(x'' - x', y'' - y')$. Thus, if (p', q'), (p'', q'') are points corresponding to (x', y'), (x'', y'') by virtue of (2), then

$$\Delta p\Delta x + \Delta q\Delta y > 0, \tag{10}$$

where $\Delta x = x'' - x'$, $\Delta p = p'' - p'$, etc. As is well known, (10) is necessary and sufficient that (4) be strictly convex; cf., e. g., [1], pp. 74-75 and p. 79.

Clearly, (10) implies the one-to-one character of the transformation (2).

Consider (8) as a function of (p, q) by virtue of the inverse of (2). Let (p_1, q), (p_2, q) be two points of $\mathfrak{D}$ and (x_1, y_1), (x_2, y_2) the corresponding points of D. Then the difference $\Delta h = h(p_2, q) - h(p_1, q)$ can be written as

$$\Delta h = x_1\Delta p + p_2\Delta x + q\Delta y - \Delta z,$$

where $\Delta p = p_2 - p_1$, $\Delta x = x_2 - x_1$, etc. The mean value theorem of differential calculus shows that $\Delta z = p^*\Delta x + q^*\Delta y$, where p^*, q^* are the partial derivatives of z at a point (x^*, y^*) on the segment joining (x_1, y_1) and (x_2, y_2). Hence

$$\Delta h = x_1\Delta p + (p_2 - p^*)\Delta x + (q - q^*)\Delta y.$$

Since the vector $(\Delta x, \Delta y)$ is a *positive* multiple of $(x_2 - x^*, y_2 - y^*)$, it follows from the case $(x', y') = (x^*, y^*)$, $(x'', y'') = (x_2, y_2)$ of (10) that $\Delta h > x_1\Delta p$. If Δh is written in the form

$$\Delta h = x_2\Delta p + p_1\Delta x + q\Delta y - \Delta z = x_2\Delta p - [(p^* - p_1)\Delta x + (q^* - q)\Delta y],$$

it follows that $\Delta h < x_2\Delta p$. Consequently, h has a partial derivative h_p with respect to p, and $h_p = x$. Similarly, h_q exists and is y. Since the inverse

$x = x(p, q)$, $y = y(p, q)$ of (2) is continuous, it follows that the partial derivatives of h are continuous.

In order to complete the proof of (II), it remains to ascertain that h is strictly convex. But this follows from (9) and (10); cf. [1], pp. 74-75 and p. 79.

REMARK. It remains undecided whether or not (II), hence a modified version of (I), remains correct if the assumption that S is strictly convex is replaced by the assumption that (2) is a one-to-one, continuous mapping of a simply connected (x, y)-domain D onto a (p, q)-domain $\mathcal{D}$. Under this assumption, when D is suitably chosen, the set of points $z_x(x, y) = \text{const.}$ is empty or a Jordan arc and the family of these Jordan arcs covers D in a *schlicht* manner. A corresponding remark applies to the family $z_y(x, y) = \text{Const.}$ If it is further assumed that the arcs $z_x(x, y) = \text{const.}$, $z_y(x, y) = \text{Const.}$ are (locally) rectifiable, then those assertions of (II) which do not refer to convexity remain correct. For then, if J is the boundary of any rectangle $c_1 \leqq p \leqq c_2$, $C_1 \leqq q \leqq C_2$ in $\mathcal{D}$, the image of J in D is rectifiable. If $x = x(p, q)$, $y = y(p, q)$ is the inverse of (2), then an integration by parts shows that

$$\int_J x\,dp + y\,dq = 0. \tag{8 bis}$$

Hence there exists on $\mathcal{D}$ a function $h = h(p, q)$ of class C^1 satisfying (9). The usual formula for obtaining h and another integration by parts show that such a function h is supplied by (8).

5. It was shown in [4], Theorem (iii), p. 368, that if $S\colon X = X(u, v)$ is a surface of class C^n, where $n \geqq 2$, and if c is a constant distinct from a pair of exceptional values c_1, c_2, then

$$Y = Y(u, v) = X + cN \tag{11}$$

is a C^{n-1}-parametrization of a surface Σ which is of class C^n. The surface $\Sigma = \Sigma_c$ is called a parallel surface of S. It was also pointed out in [4], pp. 369-370, that if S is of class C^1, then the locus of the end points of (11) need not be a surface of class C^1 (for any choice of the constant c). In what follows, there will be proved an analogue of the result of Theorem (iii) of [4], an analogue in which $n \geqq 2$ is replaced by $n = 1$, but S is assumed to be convex.

(IV) *Let $z = z(x, y)$ be convex and of class C^1 on a convex neighbor-*

hood D of $(x, y) = (0, 0)$ and let S denote the surface $X = (x, y, z(x, y))$. Let the constant c have the property that

$$(12) \qquad \alpha = x - cz_x(1 + z_x^2 + z_y^2)^{-\frac{1}{2}}, \qquad \beta = y - cz_x(1 + z_x^2 + z_y^2)^{-\frac{1}{2}}$$

is a one-to-one, continuous mapping of a sufficiently small D onto an (α, β)-domain $\mathfrak{D}$. (For instance, let $c < 0$). Then the locus of end points of $Y = (\alpha, \beta, \gamma)$, where

$$(13) \qquad \gamma = z + c(1 + z_x^2 + z_y^2)^{-\frac{1}{2}}$$

is a surface Σ of class C^1, with $Y = (\alpha, \beta, \gamma(\alpha, \beta))$ as a C^1-parametrization. (If $c < 0$, then $\gamma = \gamma(\alpha, \beta)$ is a convex function of (α, β) on every convex subdomain of $\mathfrak{D}$.)

Clearly, (11) is equivalent to (12)-(13).

It will first be verified that if $c < 0$, then (12) is a one-to-one, continuous mapping. To this end, let (12) be written as $\alpha = x - c\lambda$, $\beta = y - c\mu$, where λ, μ are defined by (2)-(3). It is readily verified from (3) that

$$(14) \qquad \Delta\lambda\Delta p + \Delta\mu\Delta q \gtrless 0 \text{ according as } (\Delta p)^2 + (\Delta q)^2 \gtrless 0,$$

where the notation is similar to that in (10); in fact, (14) is a particular case of (10), where z is the strictly convex function $z = (1 + x^2 + y^2)^{\frac{1}{2}}$ and x, y, p, q are denoted by p, q, λ, μ, respectively. The convexity (rather than strict convexity) of z in (IV) implies that

$$(15) \qquad \Delta p\Delta x + \Delta q\Delta y \geqq 0.$$

Since

$$(16) \qquad \Delta\alpha\Delta p + \Delta\beta\Delta q = (\Delta x\Delta p + \Delta y\Delta q) - c(\Delta\lambda\Delta p + \Delta\mu\Delta q),$$

it follows that $\Delta\alpha\Delta p + \Delta\beta\Delta q > 0$ if $(\Delta p)^2 + (\Delta q)^2 > 0$ and $c < 0$. On the other hand, $\Delta\alpha\Delta x + \Delta\beta\Delta y = (\Delta x)^2 + (\Delta y)^2$ if $(\Delta p)^2 + (\Delta q)^2 = 0$, by (12). Consequently, if $c < 0$, then $(\Delta\alpha)^2 + (\Delta\beta)^2 > 0$ whenever $(\Delta x)^2 + (\Delta y)^2 > 0$; so that the one-to-one character of the continuous mapping (12) follows.

It should be remarked that (12) need not be a one-to-one transformation (of some small D) for any $c > 0$, even though z is strictly convex. This is shown by an example of the type $z = g(x) + y^2$, where dg/dx is a continuous monotone function such that the sets of points where $d^2g/dx^2 = 0$ and where $d^2g/dx^2 = \infty$, respectively, cluster at $x = 0$.

As in the cases (I) and (II), it cannot be asserted that Σ is of class C^2, not even if additional smoothness conditions are imposed on S (and S is assumed to be strictly convex):

(V) *If* $z = z(x, y)$ *in* (IV) *is a polynomial which is strictly convex in the domain under consideration, and if* c *satisfies the conditions of* (IV), *then the parallel surface* $\Sigma = \Sigma_c$ *need not be of class* C^2.

In (IV), the constant c must, of course, be one of the exceptional values excluded in Theorem (iii) of [4], p. 368.

6. *Proof of* (IV). Since (12) is a one-to-one, continuous mapping of D onto $\mathcal{D}$, the function (13) can be considered as a function of (α, β) on $\mathcal{D}$ by virtue of the inverse of (12). Thus the set of points Σ which is claimed to be a surface of class C^1 has the parametrization $Y = (\alpha, \beta, \gamma(\alpha, \beta))$. Clearly, Σ is a surface of class C^1 if and only if $\gamma(\alpha, \beta)$ is a function of class C^1. It will be shown that γ is a function of class C^1 and that

$$\gamma_\alpha = z_x, \qquad \gamma_\beta = z_y. \tag{17}$$

To this end, let (α_1, β), (α_2, β) be two points of $\mathcal{D}$ and let (x_1, y_1), (x_2, y_2) be the corresponding points of D. Then the difference $\Delta\gamma = \gamma(\alpha_2, \beta) - \gamma(\alpha_1, \beta)$ can be written as

$$\Delta\gamma = \Delta z + c\Delta\nu, \quad \text{where} \quad \nu = (1 - \lambda^2 - \mu^2)^{\frac{1}{2}}; \tag{18}$$

cf. (2), (3). Also

$$\Delta\alpha = \Delta x - c\Delta\lambda, \qquad \Delta\beta = 0 = \Delta y - c\Delta\mu, \tag{19}$$

by (12). In (18), Δz can be expressed as $p^*\Delta x + q^*\Delta y$, where p^*, q^* are the partial derivatives of z at a point (x^*, y^*) of the segment joining (x_1, y_1) and (x_2, y_2). Hence

$$\Delta\gamma = p_1\Delta x + q_1\Delta y + (p^* - p_1)\Delta x + (q^* - q_1)\Delta y + c\Delta\nu,$$

where (p_1, q_1), (p_2, q_2) correspond to (x_1, y_1), (x_2, y_2) by virtue of (2). If Δx, Δy are substituted from (19) into the last formula, there results

$$\Delta\gamma = p_1\Delta\alpha + [(p^* - p_1)\Delta x + (q^* - q_1)\Delta y] + c\{p_1\Delta\lambda + q_1\Delta\mu + \Delta\nu\}. \tag{20}$$

Similarly,

$$\Delta\gamma = p_2\Delta\alpha + [(p^* - p_2)\Delta x + (q^* - q_2)\Delta y] + c\{p_2\Delta\lambda + q_2\Delta\mu + \Delta\nu\}. \tag{21}$$

By virtue of the convexity of z, the term $[\cdot\ \cdot\ \cdot]$ in (20) is non-negative when the term $[\cdot\ \cdot\ \cdot]$ in (21) is non-positive, and conversely. Similarly, the terms $\{\cdot\ \cdot\ \cdot\}$ in (20) and (21) cannot be of the same sign, since their sum, with the respective positive weight factors $(1 + p_1^2 + q_1^2)^{-\frac{1}{2}}$, $(1 + p_2^2 + q_2^2)^{-\frac{1}{2}}$,

is the scalar product of the two vectors $(\lambda_2 \pm \lambda_1, \mu_2 \pm \mu_1, \nu_2 \pm \nu_1)$, which is 0, since both $(\lambda_l, \mu_l, \nu_l)$ are of length 1. Hence (20) and (21) show that

$$\min(p_1\Delta\alpha, p_2\Delta\alpha) \leqq \Delta\gamma \leqq \max(p_1\Delta\alpha, p_2\Delta\alpha).$$

This proves the first relation in (17), and the second is proved in the same way. Since z_x, z_y are continuous functions of (x, y) and since the inverse $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$ of (12) is continuous, it follows that γ is of class C^1.

It is clear from (14), (15), (16) and (17) that $\gamma = \gamma(\alpha, \beta)$ is a convex function (on convex subdomains of $\mathfrak{D}$) when $c < 0$. This completes the proof of (IV).

7. *Proof of* (V). For small $|x|$, consider the arc

$$\Gamma\colon\ z = \tfrac{1}{2}(x^2 + \tfrac{1}{2}x^4) \tag{22}$$

in the (x, z)-plane. A surface S proving the assertion (V) will be obtained by revolving (22) about the z-axis. Clearly, the parallel surfaces of S will be obtained by revolving the parallel curves

$$\alpha = x - c(x + x^3)(1 + (x + x^3)^2)^{-\frac{1}{2}}, \qquad \gamma = z + c(1 + (x + x^3)^2)^{-\frac{1}{2}} \tag{23}$$

of (22) in the (α, γ)-plane about the γ-axis. Thus, in order to discuss the parallel surfaces of S, it is sufficient to consider the parallel curves (23) of (22).

Expansion of $(1 + (x + x^3)^2)^{-\frac{1}{2}}$ into a power series $1 - \frac{1}{2}x^2 + \cdots$ shows that the first formula in (23) is

$$\alpha = (1 - c)x - \tfrac{1}{2}cx^3 + \cdots. \tag{24}$$

On differentiating this relation,

$$d\alpha/dx = (1 - c) - 3cx^2/2 + \cdots, \tag{25}$$

it is seen that, if $c \neq 1$, then $d\alpha/dx \neq 0$ for small $|x|$ and that, if $c = 1$, then $d\alpha/dx \leqq 0$ according as $|x| \geqq 0$. Hence, for any fixed c, α is a strictly monotone function of x (for small $|x|$). This corresponds to the fact that, in the case at hand, (12) is a one-to-one mapping of (a small) D for every c.

If γ in (23) is written as a power series in x, one obtains

$$\gamma = c + \tfrac{1}{2}(1 - c)x^2 + (1/4 - 5c/8)x^4 + \cdots. \tag{26}$$

If $c \neq 1$, then the inversion of (24) gives a power series $x = \alpha/(1 - c) + \cdots$ in α. Thus the corresponding parallel curve $\gamma = \gamma(\alpha)$ of (22) becomes

$$\gamma = c + \tfrac{1}{2}\alpha^2/(1 - c) + \cdots, \qquad (c \neq 1), \tag{27}$$

by (26). If $c = 1$, then the inversion of (24) gives

$$x = x(a) = (-2a)^{1/3} + o(|a|^{1/3}),$$

as $a \to 0$, and the corresponding parallel curve has an equation of the type

$$\gamma = 1 - (3/8)(-2a)^{4/3} + o(|a|^{4/3}), \qquad (c = 1). \tag{28}$$

Since the parallel curve (28) clearly is not of class C^2, the resulting parallel surface of S, obtained by rotating (28) about the γ-axis, is not of class C^2. This proves (V).

As in the case of a sphere, the direction of the concavity of the parallel surface (cf. (27)) is reversed as c passes through a "critical value."

Appendix.*

Let Θ be a (u^1, u^2)-manifold which is topologically equivalent to the sphere $x^2 + y^2 + z^2 = 1$ and carries a metric

$$ds^2 = g_{\alpha\beta}(u^1, u^2)\, du^\alpha du^\beta, \tag{1}$$

where $\det g_{ik} > 0$, $g_{ii} > 0$, $g_{12} = g_{21}$. If the tensor g_{ik} is of class C^1 (without being of class C^2), then the existence of a continuous curvature $K = K(u^1, u^2)$ will be meant in the "integrated" sense defined by Weyl ([4], p. 43). He has stated (p. 44) the following existence theorem (along with the corresponding uniqueness theorem):

(*) Let there be given on Θ a metric tensor (g_{ik}) which is of class C^1 and such as to possess a curvature $K = K(u^1, u^2)$ which is of class C^0 (i. e., continuous) and positive on Θ. Then there exists in the X-space, where $X = (x, y, z)$, a closed convex surface $X = X(u^1, u^2)$ of class C^2 on which the metric (1) becomes the first fundamental form, that is, $ds^2 = |dX(u, v)|^2$.

When writing his classical paper [4], Weyl was well aware that it contains certain gaps. Most of these have been filled in the meantime. But as far as we know, the latter developments have failed to lead to the existence theorem denoted by (*) above, and the following remarks would indicate the possibility that (*) is false. The situation is as follows:

For any positive integer n, let $(*_n)$ denote the statement which results if C^1 and C^0 in the assumptions, and C^2 in assertion, of (*) are replaced by the respective classes C^n, C^{n-1}, C^{n+1} (so that (*) is identical with the case $n = 1$ of $(*_n)$). We shall prove that $(*_2)$ *is false, as is* $(*_n)$ *for every* $n > 1$.

* Added March 30, 1953.

Weyl's case, $n = 1$, will remain undecided. Only the case $n = 2$ will be considered, since it will be clear from the proof that the method applies to $n = 3, 4, \cdots$ also.

In order to disprove $(*_2)$, use will be made of the following fact: As shown in [2], p. 135 (bottom), there exists on a circle $0 \leqq x^2 + y^2 < a^2$ a function $z = z(x, y)$ of class C^2 which is regular analytic for $0 < x^2 + y^2 < a^2$, ceases to be of class C^3 at $(x, y) = (0, 0)$, its first derivatives satisfy

$$p = 0 \text{ and } q = 0 \text{ at } (x, y) = (0, 0), \tag{2}$$

its third derivatives are subject to

$$r_x, s_x, t_x, r_y, s_y, t_y = O(|\log(x^2 + y^2)|) \text{ if } 0 < x^2 + y^2 \to 0, \tag{3}$$

finally $K(x, y) = (rt - s^2)/(1 + p^2 + q^2)^2$, that is, the curvature of the piece of surface $S: z = z(x, y)$, where $0 \leqq x^2 + y^2 < a^2$, is positive and of class C^1 (i. e., K_x, K_y exist and are continuous at $(x, y) = (0, 0)$ also).

Since the metric (1) on S is

$$ds^2 = (1 + p^2)dx^2 + 2pqdxdy + (1 + q^2)dy^2, \tag{4}$$

where $(x, y) = (u^1, u^2)$ and $g_{11} = 1 + p^2, \cdots$, the metric tensor (g_{ik}) is of class C^1 in (x, y), simply because the function $z(x, y)$ is of class C^2. But it will be shown that, despite the fact that $z(x, y)$ is not of class C^3, the metric tensor (g_{ik}) is of class C^2 in (x, y).

First, the function $g_{11}(x, y)$, being the coefficient of dx^2 in (4), has the first partial derivatives $2pr$, $2ps$ for $0 \leqq x^2 + y^2 < a^2$ and the second partial derivatives

$$2r^2 + 2pr_x, \qquad 2rs + 2pr_y, \qquad 2s^2 + 2ps_y$$

for $0 < x^2 + y^2 < a^2$. Since (2) implies that $p(x, y)$ and $q(x, y)$ are $O(x^2 + y^2)^{\frac{1}{2}}$ as $x^2 + y^2 \to 0$, it follows from (3) and from the last formula line that the second partial derivatives of g_{11} are

$$\text{const.} + O(x^2 + y^2)^{\frac{1}{2}} \log(x^2 + y^2),$$

and therefore const. $+ o(1)$. In view of well-known properties of derivatives which tend to limits, this proves that $g_{11} = g_{11}(x, y)$ has continuous second partial derivatives at $(x, y) = (0, 0)$. Since the same follows if $g_{11} = 1 + p^2$ is replaced by $g_{12} = pq$ or $g_{22} = 1 + q^2$, the metric tensor (g_{ik}) is of class C^2 in (x, y) for $0 \leqq x^2 + y^2 < a^2$.

If the $a > 0$ occurring in the definition of S is replaced by any smaller positive number, then S remains analytic on its boundary curve. It is clear from the properties of S that it is possible to choose a closed convex surface,

say U, which contains S, is of class C^∞ except at the point $(x, y) = (0, 0)$, and satisfies $K \neq 0$. Then U is an ovaloid of class C^2 on which K is positive and of class C^1 (in fact, this is true of K at the point $(x, y) = (0, 0)$ of S also). The metric tensor on U, being of class C^2 at every point of S, is of class C^2 on U. Hence, if $(*_2)$ is granted, its application to the metric tensor of U supplies the existence of an ovaloid, say V, which is of class C^3 and carries the same metric as U. Since both V and U are of class C^2 and of positive curvature, it now follows from the C^2-extension ([1], pp. 60-72 (A. D. Alexandrov); cf. [5], pp. 206-213) of Herglotz's C^3-theorem [3] that U and V are congruent. Consequently, U is of class C^3. But this is a contradiction, since the portion S of U is not of class C^3 at its point $(x, y) = (0, 0)$.

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ON ASYMPTOTIC PARAMETRIZATIONS.*

By PHILIP HARTMAN and AUREL WINTNER.

In what follows, the (sufficiently small) surface S will be of class C^n for some $n \geqq 2$, and it will be assumed that *the Gaussian curvature K is negative on S*. Thus, if $X = X(u^1, u^2)$ is a C^n-parametrization of S, and if $h_{ik}du^i du^k$, where $h_{ik} = h_{ik}(u^1, u^2)$, denotes the second fundamental form of S in this parametrization, then

$$h_{ik}(u^1, u^2)\, du^i du^k = 0 \tag{1}$$

determines a unique pair of distinct directions $du^1 : du^2$, the asymptotic directions, through every point (u^1, u^2) (since

$$\det h_{ik} < 0 \tag{2}$$

in view of $K < 0$), and the three functions $h_{ik}(u^1, u^2)$ are of class C^{n-2}, as is the function $K(u^1, u^2)$.

If the parametrization $S : X = X(u^1, u^2)$ is such as to make both functions $h_{ii}(u^1, u^2)$ vanish identically, that is, if both $u^1 = \text{const.}$ and $u^2 = \text{Const.}$ are solutions of (1) for arbitrary values of the constants, then $X = X(u^1, u^2)$ will be called an asymptotic parametrization of S. In this terminology, Corollary 1 in [2] can be stated as follows:

(i) *If S possesses an asymptotic C^n-parametrization, where $n \geqq 2$, then it possesses C^{n+1}-parametrizations.*

The following considerations deal with the converse of the question answered by this theorem, in the sense that the existence of an *asymptotic* parametrization of a specified degree of smoothness will be discussed under the assumption that a given degree of smoothness is assumed in *some* parametrization. All that is straightforward in this direction is the following fact:

(ii) *If S possesses a C^m-parametrization, where $m \geqq 3$, then it possesses asymptotic C^{m-2}-parametrizations.*

Remark. If $m = 3$ (that is, if the asymptotic parametrization which is claimed by (ii) is not of class C^2), then the coefficient functions h_{ik} of (1)

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belonging to the C^{m-2}-parametrization cannot be calculated directly and must be meant as having been determined first in the C^m-parametrization and then transformed, by means of the rule of doubly covariant tensors and a C^1-transformation $(u^1, u^2) \to (v^1, v^2)$ of positive Jacobian, to the asymptotic parameters. A corresponding proviso holds for $m = 2$ in (iii) and (iv), below.

Proof of (ii). Let $X = X(u^1, u^2)$, where

$$(u^1)^2 + (u^2)^2 < a^2, \tag{3}$$

be a C^m-parametrization of S near a point, say the point $X = (0, 0, 0)$, where $X = (x, y, z)$. It is clear from (2) that, after a fixed rotation of the coordinate system (x, y, z) about $(0, 0, 0)$, it can be supposed that $h_{11}(u^1, u^2) \neq 0$ holds at $(u^1, u^2) = (0, 0)$, and therefore on the domain (3) if a is small enough. Then, if

$$f_j = (-h_{12} + (-1)^j \mid \det h_{ik} \mid^{\frac{1}{2}})/h_{11}, \qquad \text{where } j = 1, 2, \tag{4}$$

the quadratic equation (1) splits into the two differential equations

$$(5_1) \quad du^1/du^2 = f_1(u^1, u^2), \qquad (5_2) \quad du^1/du^2 = f_2(u^1, u^2),$$

where both functions (4) are of class C^{m-2} on the domain (3), since all three functions $h_{ik}(u^1, u^2)$ are. But the assumption $m \geqq 3$ means that the index of the class C^{m-2} is positive. It follows therefore from a standard theorem that, if a in (3) is small enough, the differential equations (5_1), (5_2) possess on the domain (3) first integrals, say

$$v^1(u^1, u^2) = v^1 = \text{const.}, \qquad v^2(u^1, u^2) = v^2 = \text{Const.}, \tag{6}$$

such that the functions (6) of (u^1, u^2) are of class C^{m-2} and have a Jacobian $\partial(v^1, v^2)/\partial(u^1, u^2)$ which becomes $1 \neq 0$ at $(u^1, u^2) = (0, 0)$. Consequently, the substitution (6) has a local inverse, say

$$u^1 = u^1(v^1, v^2), \qquad u^2 = u^2(v^1, v^2), \tag{7}$$

where both functions $u^j(v^1, v^2)$ are of class C^{m-2}. Accordingly, there results a local C^{m-2}-parametrization of S in terms of (v^1, v^2) if (7) is substituted into the given parametrization $X = X(u^1, u^2)$ of S. Since (6) means that the parameter curves v_1, v_2 are asymptotic curves, the proof of (ii) is complete.

It will be noted that (ii) claims less than the converse of the assertion of (i) (when $n \geqq 3$), since the difference of the two C-indices is 1 in (i) but 2 in (ii). Hence the following theorem, in which the C^{m-2} of (ii) is improved to C^{m-1} (and, incidentally, the $m \geqq 3$ of (ii) to $m \geqq 2$), is not without interest.

(iii) *If S possesses a C^m-parametrization, where $m \geqq 2$, and if the curvature $K(< 0)$ is a function of class C^{m-1} in this parametrization, then S possesses asymptotic C^{m-1}-parametrizations.*

The proof of this theorem will be omitted, since it differs from the proof of Theorems (iii), (iv) in [1] (pp. 152-153, 156) only in obvious details, the "Legendre transformation" to be used being the same as loc. cit.

Although the difference of the C-indices is the same ($= 1$) in (i) as in (iii), the latter theorem is not a straightforward converse of the former, since, under the assumptions of (i), the function K is of class C^{n-2} only; so that the last assumption of (iii) imposes on K the restriction of an additional degree of continuous differentiability. The above proof of (ii) indicates that this (or some other) additional restriction cannot be omitted, that is, that the straightforward converse of (i) is false. We shall prove this negation only for the case $m = 2$ (excluded in (ii)), as follows:

(iv) *Suppose that S possesses a C^m-parametrization and an asymptotic C^{m-2}-parametrization. Then, if $m = 2$, it need not possess any asymptotic C^{m-1}-parametrization.*

Note that if $m > 2$, then the first assumption of (iv) implies the second by virtue of (iii).

Needless to say, the second assumption of (iv), that requiring the existence of an asymptotic C^0-parametrization ($C^{m-2} = C^0$), is meant in the following sense: All asymptotic curves of S together form two families each of which is a continuous family of Jordan arcs which cover S in a *schlicht* manner (the arcs are, incidentally, curves of class C^1, since they are defined by (5)). Thus it is *assumed* that every point of S should issue just one asymptotic curve in an asymptotic direction. That this is actually an assumption, was proved in [1], pp. 153-156, by constructing a suitable S of class C^2.

Actually, the example given loc. cit. happens to be such that one family of asymptotic curves fails to contain a continuous family of curves which covers S in a *schlicht* manner (cf. [1], the last paragraph of Section 4, p. 155). Hence *if $m \geqq 3$ in* (ii) *is replaced by $m = 2$, then no interpretation of the assertion of* (ii) *remains true.*

Proof of (iv). Let $G(x)$, $H(y)$ be functions possessing continuous second derivatives

$$g(x) = d^2G/dx^2, \qquad h(y) = d^2H/dy^2 \tag{8}$$

on respective intervals $|x| \leqq$ const., $|y| \leqq$ Const., and let

$$g(0) = 0, \qquad h(0) = 0. \tag{9}$$

Then, as in [1], p. 153, the surface

$$S\colon\ z = z(x, y), \text{ where } z(x, y) = G(x) + H(y) + xy, \tag{10}$$

is of class C^2 and its Gaussian curvature $K = K(x, y)$, being -1 at $(x, y) = (0, 0)$, is negative when (x, y) is in a small rectangle containing $(0, 0)$. On such a rectangle, one of the differential equations (5_1)-(5_2) of the asymptotic curves of (10) is

$$y' = f(x, y), \text{ with } y' = dy/dx, \tag{11}$$

where, as in [1], p. 153,

$$f = \{-1 + (1 - gh)^{\frac{1}{2}}\}/h \text{ if } h \neq 0 \text{ and } f = -\tfrac{1}{2}g \text{ if } h = 0, \tag{12}$$

and, correspondingly, the other of the differential equations (5_1)-(5_2) is equivalent to

$$x' = f^*(x, y), \text{ with } x' = dx/dy, \tag{11 bis}$$

where

$$f^* = \{-1 + (1 - gh)^{\frac{1}{2}}\}/g \text{ if } g \neq 0 \text{ and } f^* = -\tfrac{1}{2}h \text{ if } g = 0. \tag{12 bis}$$

It is clear from (12), (12 bis) that both functions f, f^* of (x, y) are continuous. Note that the second case of (12) or (12 bis) occurs, by (9), at $(0, 0)$ but possibly also at points (x, y) which cluster at $(0, 0)$.

The functions $g(x)$, $h(y)$ will be chosen below so that the solutions of (11), (11 bis) are uniquely determined by their initial conditions. For small $|x|, |y|, |u|, |v|$, let

$$x = x(y, u), \qquad y = y(x, v) \tag{13}$$

be the solutions of (11 bis), (11), respectively, satisfying

$$x(0, u) = u, \qquad y(0, v) = v. \tag{14}$$

Both functions $x(y, u)$, $y(x, v)$ are continuous functions of two variables and the substitution (13) has a unique continuous inverse

$$u = u(x, y), \qquad v = v(x, y) \tag{15}$$

near the origin (in fact, if $y = y(x; u, v)$ is the solution of (11) satisfying $y(u; u, v) = v$, then $v(x, y) = y(0; x, y)$). Thus (15) defines a one-to-one, continuous transformation of a vicinity of $(x, y) = (0, 0)$ into a vicinity of $(u, v) = (0, 0)$. Hence, if the inverse of (15), say

$$x = x(u, v), \qquad y = y(u, v), \tag{15 bis}$$

is substituted into (10), there results for the surface S an asymptotic C^0-parametrization, say $X = X(u, v)$, in a vicinity of the point $(u, v) = (0, 0)$.

The function $y(x, v)$ occurring in (13) has, by (11), a continuous partial derivative with respect to x, namely, $f(x, y(x, v))$. The functions $g(x)$, $h(y)$ will be chosen below in such a way that $y(x, v)$ will fail to have a continuous partial derivative with respect to v at $(x, v) = (0, 0)$. Then, since the second relation in (14) implies that $y_v(0, v) = 1$, it will follow that $v = v(x, y)$ in (15) is not of class C^1 in any vicinity of $(x, y) = (0, 0)$. Consequently, $X = X(u, v)$ is not a C^1-parametrization of S (in any vicinity of $(x, y) = (u, v) = (0, 0)$). For otherwise the transformation (15), the inverse of (15 bis), were of class C^1, since, by virtue of (10), the tangent plane of S is not perpendicular to the (x, y)-plane.

If $X = X(s; t)$ is any asymptotic C^0-parametrization of S, then it results from $X = X(u, v)$ by a transformation of the form

$$s = s(u), \qquad t = t(v), \tag{16}$$

where $s(u)$, $t(v)$ are continuous, strictly monotone functions for small $|u|$, $|v|$, respectively, and it can be supposed that $s(0) = 0$, $t(0) = 0$. No such parametrization $X(s; t) = (x(s; t), y(s; t), z(s; t))$ can be of class C^1 on any vicinity of $(s; t) = (0, 0)$. For otherwise the transformation $x = x(s; t)$, $y = y(s; t)$ has a C^1-inverse, $s = s(x, y)$, $t = t(x, y)$, with a non-vanishing Jacobian at $(x, y) = (0, 0)$. This inverse is given, in terms of (15) and the functions (16), by

$$s = s(u(x, y)), \qquad t = t(v(x, y)). \tag{16 bis}$$

At $y = 0$ and $x = 0$, respectively, this becomes $s = s(x, 0) = s(x)$ and $t = t(0, y) = t(y)$. Hence the monotone functions (16) have non-vanishing continuous derivatives. But this is impossible, since (15) is not, but (16 bis) is, a C^1-transformation.

Thus, in order to complete the proof of (iv), it is sufficient to exhibit a pair of continuous functions $g(x)$, $h(y)$ (for small $|x|$, $|y|$, respectively) satisfying (9) and having the properties that the solutions of (11) and (11 bis) are uniquely determined by initial conditions and that $y = y(x, v)$ in (13) does not have a continuous partial derivative y_v in any vicinity of $(x, v) = (0, 0)$.

To this end, let p, q be numbers satisfying

$$0 < q < \tfrac{1}{2} \text{ and } p = q/(1 + 3q). \tag{16}$$

Choose

$$h(y) = 0 \text{ if } y \leqq 0 \text{ and } h(y) = y^p \text{ if } y > 0. \tag{17}$$

Put

$$a_k = 2\pi \sum_{j=k+1}^{\infty} j^{-2} \text{ for } k = 0, 1, \cdots \tag{18}$$

and let

$$M > 1 \tag{19}$$

be a fixed constant. In terms of q and M, define $g(x)$ as follows:

$$\begin{aligned} &g(x) = 0 \text{ if } x \leqq 0, \\ &g(x) = Mk^{-q} \sin^2 k^2(t-a) \text{ if } a < t \leqq a + k^{-2}\pi, \\ &g(x) = -M(k^{-q} - k^{-3q})\sin^2 k^2(t-a) \text{ if } a + k^{-2}\pi < t \leqq a + 2k^{-2}\pi, \end{aligned} \tag{20}$$

where $a = a_k$ and $k = 1, 2, \cdots$.

Clearly, $g(x)$ and $h(y)$ are continuous (say, for $|x| \leqq a_0$ and $|y| \leqq 1$, respectively) and satisfy (9). Since (18) implies that

$$a_k \sim \text{const. } k^{-1} \text{ as } k \to \infty \tag{21}$$

holds for a positive constant,

$$g(x) = O(x^q) \text{ as } x \to 0. \tag{22}$$

Furthermore, if $a = a_k$, then, as $k \to \infty$,

$$\int_0^a g\,dt = \text{const. } M \sum_{j=k+1}^{\infty} k^{-3q}k^{-2} \sim \text{Const. } Mk^{-1-3q},$$

$$\int_0^a g^2\,dt = \text{const. } M^2 \sum_{j=k+1}^{\infty} k^{-2q}k^{-2} \sim \text{Const. } M^2k^{-1-2q}.$$

Also, if $(a =)a_k < x < a_{k-1}$, then

$$\int_a^x g\,dt = O(k^{-q-2}) \text{ and } \int_a^x g^2\,dt = O(k^{-2q-2}).$$

It follows therefore from (21), and from the assumptions on q in (16), that there exist positive constants C_1, C_2 (independent of M) such that, as $x \to 0$,

$$\int_0^x g\,dt \sim C_1 M x^{1+3q}, \tag{23}$$

$$\int_0^x g^2\,dt \sim C_2 M^2 x^{1+2q}. \tag{24}$$

For small $|x|$, $|y|$, let (11) be written as

$$y' = -\tfrac{1}{2}g + g^2h/8 + \cdots, \text{ where } y' = d/dx, \tag{25}$$

$g = g(x)$, $h = h(y)$ (the right-hand side of (25) is g times a power series in gh, obtained from the expansion of the $(1 - gh)^{\frac{1}{2}}$ in f).

For small $|x_0|$, $|y_0|$, any initial condition

$$y(x_0) = y_0 \tag{26}$$

determines a unique solution of (11) (and/or (25)) if $x_0 < 0$, since $f(x, y) \equiv 0$ when $x < 0$; cf. the first part of (20). If $y_0 \neq 0$, then (26) determines a unique solution of (11), since $f(x, y)$ possesses a continuous partial derivative with respect to y for $y \neq 0$. If $y_0 = 0$ and $g(x_0) \neq 0$, then (11) is equivalent, for (x, y) near (x_0, y_0), to a differential equation of the type (11 bis) in which $f^*(x, y)$ possesses a continuous partial derivative with respect to x. Thus there only remains to consider the cases $y_0 = 0$, $g(x_0) = 0$.

If $x_0 = y_0 = 0$, a solution of (25) and (26) satisfies $y' = O(x^q)$, by (22). Hence $y(x) = O(x^{1+q})$ as $x \to 0$. It then follows from (20), (22) and (23) that

$$y(x) \equiv 0 \text{ if } x \leqq 0 \text{ and } y(x) \sim -\tfrac{1}{2}C_1 M x^{1+3q} \text{ as } x \to +0.$$

Thus $y(x) \leqq 0$ for small $|x|$, and so, by (17),

$$y(x) = -\tfrac{1}{2}\int_0^x g\,dt \quad (\leqq 0) \tag{27}$$

is the unique solution of (25) satisfying $y(0) = 0$.

If $g(x_0) = 0$ but $x_0 > 0$, then $|g(x)|/(x - x_0)^2$ is between two positive constants for x near x_0; cf. (20). It follows that any solution of (25) and $y(x_0) = 0$ has the property that $|y(x)|/|x - x_0|^3$ is between two positive constants for x near x_0. If $y = y_1(x)$ and $y = y_2(x)$ are two such solutions, it can be supposed that $|y_1(x)| \leqq |y_2(x)|$. Since

$$h(y_2) - h(y_1) = O(y_2 - y_1)/|y_1|^{1-p},$$

it follows that

$$(y_2 - y_1)' = O(g^2(h(y_2) - h(y_1))) = O((y_2 - y_1)|x - x_0|^{4-3(1-p)})$$

as $x \to x_0$. Since $4 - 3(1 - p) > 0 > -1$ when $0 < p < 1$, standard uniqueness theorems (for example, Osgood's or Nagumo's criterion) imply that $y_2 \equiv y_1$.

Thus, when $|x_0|$, $|y_0|$ are small, (26) determines (locally) a unique solution of (11) and/or (25).

Let (11 bis) be written as

$$x' = -\tfrac{1}{2}h + h^2 g/8 + \cdots, \text{ with } x' = dx/dy, \tag{28}$$

where $|x|$, $|y|$, are small. For small $|x_0|$, $|y_0|$, consider the initial condition

$$x(y_0) = x_0. \tag{29}$$

This determines a unique solution of (11 bis) and/or (28) if $x_0 \neq 0$, since $f^*(x, y)$ in (11 bis) possesses a continuous partial derivative with respect to x if $x \neq 0$. If $x_0 = 0$ and $y_0 \neq 0$, then (11 bis) can be replaced, for (x, y) near (x_0, y_0), by a differential equation (11) in which $f(x, y)$ has a continuous partial derivative with respect to y. There remains therefore only the case $x_0 = y_0 = 0$. But then (28) and (29) have the unique solution $x = x(y)$ given by

$$x(y) = 0 \text{ if } y \leqq 0 \text{ and } x(y) = -\tfrac{1}{2}\int_0^y h\,dt \ (< 0) \text{ if } y > 0;$$

cf. the first parts of (17) and (20). Consequently, every initial condition (29) determines (locally) a unique solution of (11 bis).

In order to complete the proof of (iv), it remains to be shown that if $y = y(x, v)$ is the solution of (11) and/or (25) satisfying $y(0, v) = v$, then $y(x, v)$ is not of class C^1 on any vicinity of $(x, v) = (0, 0)$. It is clear that $y(x, v)$ is of class C^1 on every (x, v)-domain on which $y(x, v) \neq 0$. On such a domain, $y(x, v)$ has a continuous second mixed partial derivative, $y_{xv} = y_{vx}$, which can be calculated directly from (11) and/or (25).

Suppose that $y(x, v)$ is of class C^1 on a vicinity of $(x, v) = (0, 0)$. Then $y_v(x, v) = 1 + o(1)$ as $(x, v) \to (0, 0)$, since $y_v(0, v) \equiv 1$. Hence, when $y(t, v) \neq 0$ for $0 \leqq t \leqq x$, integration of y_{xv} with respect to x gives

$$y_v(x, v) = 1 + (8^{-1} + o(1))\int_0^x g^2(t) h_y(y(t, v))\,dt \tag{30}$$

for small positive x and v.

From (25),

$$y(x, v) = v - \tfrac{1}{2}\int_0^x g\,dt + (8^{-1} + o(1))\int_0^x g^2 h\,dt, \tag{31}$$

as $(x, v) \to (+0, +0)$. For small $v > 0$, let $x = x(v)$ be determined by the equation

$$16^{-1}(2v)^p \int_0^x g^2\,dt = v. \tag{32}$$

Then, according to (24), $x(v)$ satisfies $(1/16)(2v)^p C_2 M^2 x^{1+2q} \sim v$; so that there exists a constant $A > 0$ (independent of M) satisfying

$$x(v) \sim AM^{-2/(1+2q)} v^{(1-p)/(1+2q)}.$$

The definition (16) of p shows that the exponent of v is $1/(1+3q)$. Thus

$$x(v) \sim AM^{-2/(1+2q)} v^{1/(1+3q)}. \tag{33}$$

It is clear from (23), (31) and (32) that

$$y(x, v) \leqq 2v \text{ if } 0 \leqq x \leqq x(v). \tag{34}$$

Furthermore, for $x \leqq x(v)$, it follows from (23) that

$$\tfrac{1}{2} \int_0^x g\,dt = \tfrac{1}{2}(1 + o(1)) C_1 M x^{1+3q} \leqq \tfrac{1}{2}(1 + o(1)) C_1 A^{1+3q} M^{1-2(1+3q)/(1+2q)} v.$$

Since C_1 and A do not depend on M and since the last exponent of M is negative, it is clear that, if M is sufficiently large,

$$\tfrac{1}{2} \int_0^x g\,dt < v \text{ for } 0 \leqq x \leqq x(v).$$

Hence (31) shows that $y(t, v) > 0$ for $0 \leqq t \leqq x(v)$, and so (30) is valid for $x = x(v)$. It follows from (30) and (34) that, for $x = x(v)$,

$$y_v(x, v) \geqq 1 + (8^{-1} + o(1)) p (2v)^{p-1} \int_0^x g^2 dt$$

as $v \to +0$ and therefore, by (32),

$$y_v(x(v), v) \geqq 1 + p + o(1),$$

where $p > 0$. But this contains a contradiction, since $y_v(x, v) = 1 + o(1)$ as $(x, v) \to (0, 0)$.

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ESSENTIAL SETS AND FIXED POINTS.*

By BARRETT O'NEILL.

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1. Introduction. Let X be a topological space and let X^X be the space of (continuous) self-mappings of X furnished with the compact-open topology. M. K. Fort [1] calls a fixed point $x \in X$ of $f \in X^X$ *essential* if each map sufficiently near f has a fixed point arbitrarily near x. Fort investigated the questions of existence and recognition of fixed points in various cases, always requiring that X have the fixed point property. We continue this program with three changes: first, instead of essential *points* we consider essential *sets*; second, we search for essential sets in a specified region of X; third, we drop the requirement that X have the fixed point property.

(1.1) *A subset S of X is* essential *with respect to $f \in X^X$ if given any neighborhood V of S there is a neighborhood N of f in the compact-open topology on X^X such that each map in N has a fixed point in V.*

If an essential set reduces to a single point, then that point is fixed and is essential in the former sense. The fixed point property means that X itself is essential with respect to all its self-maps. The general problem is to determine all the essential sets of a given map $f \in X^X$. In particular we consider these two questions:

Location. Given an open set U, does U contain an essential component of the set $F(f)$ of fixed points of f?

Recognition. Given a component K of $F(f)$, is K essential?

Such questions cannot be settled by a knowledge of the essential points of f. In fact a component K of $F(f)$ isolated in a Euclidean neighborhood never contains an essential point, even though K itself is essential.

We make use of a "fixed point index" for sets similar to that of Leray [2] and others. An elementary proof is given of the existence of a suitable index when X is a finite polyhedron. Actually, using Čech theory, our methods apply to non-polyhedral spaces which are locally connected in the sense of homology.

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2. The trace function. We follow the exposition of cohomology theory in [3] with a few minor changes. Let X be a finite polyhedron, and let T denote any triangulation of X obtained by subdividing the basic triangulation of X. Let $C(T)$ be the vector space of alternate cochains on T with real coefficients. For each simplex $\sigma_i \in T$ designate one of its two orientations as positive, and again denote the positively oriented simplex by σ_i. Let s_i be the (elementary) cochain such that $s_i(\sigma_j) = \delta_{ij}$. Define an inner product on $C(T)$ by the formula $(s_i, s_j) = \delta_{ij}$. Let n_i be the dimension of s_i.

If $f \in X^X$, f determines a class of cochain transformations of $C(T)$ into itself as follows: If T' is a sufficiently fine triangulation of X, there is a simplicial approximation of f inducing a cochain transformation $f' : C(T) \to C(T')$. If we follow f' by the usual subdivision transformation $\pi' : C(T') \to C(T)$, the composition function $f^+ = \pi' f'$ is called a cochain *transformation* induced by f.

If $f \in X^X$, U is an open set with no fixed points of f on its boundary, and f^+ is a cochain transformation on $C(T)$ induced by f, we define the *trace* of f^+ on U to be the integer $L(f^+, U) = \sum(-1)^{n_i}(s_i, f^+(s_i))$, where the summation extends over those elementary cochains s_i of $C(T)$ for which $\sigma_i \cap U \neq \phi$. Thus if T is sufficiently fine, we have $L(f^+, X) = L(f)$, the Lefschetz number of f.

Let F be a closed set of X and let $f \in X^X$. If $\{U_1, \cdots, U_k\}$ is a collection of open sets of X such that $\bar{U}_i \cap \bar{U}_j = \phi$ if $i \neq j$, the boundary of U_i contains no fixed points of f, and $F \subset \bigcup U_i$, then we say that $\{U_1, \cdots, U_k\}$ *partitions* F.

If T is sufficiently fine each of its simplexes contributes to the trace on at most one open set U_i. Thus we have

LEMMA 2.1. *Let $\{U_1, \cdots, U_k\}$ partition the set $F(f)$ of fixed points of $f \in X^X$. If the triangulation T of X is sufficiently fine, and f^+ is a cochain transformation induced by f, then* $\sum_{i=1}^{k} L(f^+, U_i) = L(f)$.

It remains to show for any such U_i that $L(f^+, U_i)$ is independent of the choice of f^+ and T (provided the latter is sufficiently fine). The main step is the following elementary extension theorem:

LEMMA 2.2. *Let $f \in X^X$ and let f_i $(i = 1, 2)$ be a simplicial approximation of f from triangulations T_i' to T_i. Let B_i be a T_i-subpolyhedron of X and A_i a T_i'-subpolyhedron such that* a) $\mathrm{Cl\,St}\, B_1 \cap \mathrm{Cl\,St}\, B_2 = \phi$ *and* b) $f_i(A_i) \subset B_i$. *Then $f_1 \mid A_1$ and $f_2 \mid A_2$ have a common extension $f' \in X^X$ which is a simplicial approximation of f from a triangulation T' to T.*

Proof. Let $C_i = \mathrm{Cl\ St}\, B_i$.[1] Since all triangulations are subdivisions of a common triangulation, we may extend $T_1 \mid C_1$ and $T_2 \mid C_2$ to a triangulation T. Similarly, extend $T_1' \mid \mathrm{Cl\ St}\, A_1$ and $T_2' \mid \mathrm{Cl\ St}\, A_2$ to a triangulation T', and let T^j denote the j-th barycentric subdivision of T' modulo $A_1 \cup A_2$ (that is, modify the usual subdivision operation by adding no new vertices to $A_1 \cup A_2$). T^j is also an extension of $T_1' \mid A_1$ and $T_2' \mid A_2$. If T^j is sufficiently fine and the vertex v is not in $\mathrm{St}\, A_i$ then $\mathrm{St}(v)$ has arbitrarily small diameter. Extend $T_1 \mid B_1$ and $T_2 \mid B_2$ to a triangulation T. Choose T^j so fine that if v is one of its vertices in $X\text{-}(\mathrm{St}\, A_1 \cup \mathrm{St}\, A_2)$, then $\mathrm{St}(v) \subset f^{-1}(St(w))$ for some vertex w of T. Let $f'v = w$. If $v \in A_i$, let $f'(v) = f_i(v)$. Finally if $v \in \mathrm{St}\, A_i - A_i$, there is a vertex v' of T_i such that $\mathrm{St}(v) \subset \mathrm{St}(v')$. Let $f'(v) = v'$. The vertex assignment above determines the desired simplicial approximation of f.

It is now easy to prove

LEMMA 2.3. *If U_1 is an open set of X with no fixed points of f on its boundary then the trace $L(f^+, U_1)$ is independent of the choice of f^+ and T, provided T is sufficiently fine.*

Proof. Let U_2 be another open set such that $\{U_1, U_2\}$ partitions the set $F(f)$ of fixed points of f. Let f_i be a simplicial approximation of f from T_i' to T_i $(i = 1, 2)$. We must prove $L(f_1^+, U_1) = L(f_2^+, U_1)$.

Let $B_i = \mathrm{Cl\ St}\, U_i$ and let $A_i = f_i^{-1}(B_i)$. Then 2.2 applies, so f has a simplicial approximation g from T' to T extending $f_1 \mid A_1$ and $f_2 \mid A_2$. But this implies that g^+ may be chosen so that

$$L(g^+, U_1) = L(f_1^+, U_1) \quad \text{and} \quad L(g^+, U_2) = L(f_2^+, U_2).$$

If T is sufficiently fine we may combine these equations with the result of 2.2:

$$L(f_2^+, U_1) + L(f_2^+, U_2) = L(g^+, U_1) + L(g^+, U_2)$$

to obtain $L(f_1^+, U_1) = L(f_2^+, U_1)$.

This lemma permits the following definition:

(2.4) *If U is an open set of X with no fixed points of $f \in X^X$ on its boundary, the* trace *$L(f, U)$ of f on U is the common value of $L(f^+, U)$ for all sufficiently fine triangulations T.*

We collect the fundamental properties of the trace function in

[1] The operators Cl and St refer to the triangulation with which their argument is associated.

THEOREM 2.5. *Let U and V be open sets of X whose boundaries contain no fixed points of $f \in X^X$. Then the trace function L has the properties:*

L1. *If U contains no fixed points of f, then $L(f, U) = 0$.*

L2. $L(f, U) + L(f, V) = L(f, U \cup V) + L(f, U \cap V)$.

L3. *There is a neighborhood N of f such that $g \in N$ implies (g, U) admissible and $L(f, U) = L(g, U)$.*

L4. *If $\bar{U}$ is a subpolyhedron of X and $f(U) \subset U$, then $L(f, U) = L(f \mid \bar{U})$, the Lefschetz number of $f \mid \bar{U}$.*

L5. *If $U \cup f(U)$ is contained in a subpolyhedron of X isomorphic under a map h to a subpolyhedron of the finite polyhedron Y and if $g \in Y^Y$ is such that $gh = hf$ on U, then $L(f, U) = L(g, h(U))$.*

Proof. Properties L145 follow easily from the definition of L. It suffices to prove L2 in case $U \cap V = \phi$, for this result may then be applied to the sets $U - V$, $U \cap V$ and $V - U$. So suppose $U \cap V = \phi$ and let W be a third set such that $\{U, V, W\}$ partitions $F(f)$. Then both $\{U \cup V, W\}$ and $\{U, V, W\}$ partition $F(f)$, hence by 2.2 we have

$$L(f, U) + L(f, V) + L(f, W) = L(f, U \cup V) + L(f, W)$$

and the result follows.

To prove L3, note that in this case maps g such that (g, U) is admissible form a neighborhood of f in X^X. It follows from 2.1 and 2.3 that for any sufficiently fine triangulation T there is a neighborhood N of f such that if $g \in N$ then $L(g, U) = L(g, U)$ for any choice of g. Finally, for every T there is a neighborhood P of f such that the maps in P have a common simplicial approximation and hence a common induced cochain transformation g. Choose T as above, then if $g \in N \cap P$ we have

$$L(g, A) = L(g^+, A) = L(f, A).$$

In the final section we prove that L is uniquely determined on the class of finite polyhedra by properties L2345.

3. Admissible sets. Let X be a space and L an integer-valued function of pairs (f, U) where U is an open set of X with no fixed points of f on its boundary. We call L a *fixed point index on X* if L has the properties L1234 of Theorem 2.5. In this and the following section we assume that X is a space with a fixed point index L. The following properties of L are easily proved.

(3.1) *If $L(f, U) \neq 0$, then U is essential.* This is our basic criterion for the essentiality of U.

(3.2) *If $L(f) \neq 0$ then at least one set in any partition of $F(f)$ is essential.*

(3.3) *If f_t is a homotopy such that for each $t \in [0, 1]$ no fixed points appear on the boundary of U, then $L(f_0, U) = L(f_1, U)$.*

In fact by L1, L is constant on components of the open subspace of X^X consisting of maps for which the boundary of U is free of fixed points.

(3.4) *If the symmetric difference of U and V contains no fixed points of f, then $L(f, U) = L(f, V)$.*

(3.5) *If $\bar{V}$ is a subpolyhedron of X such that $\{U, V\}$ partitions $F(f)$ and $f(V) \subset V$, then $L(f, U) = L(f) - L(f \mid \bar{V})$.*

This fact is particularly useful since it characterizes $L(f, U)$ in terms of *homological* invariants. One would like such a characterization for arbitrary pairs (f, U).

If $A \subset X$ and $f \in X^X$ let $F(f, A)$ denote the set of fixed points of f in A. We say that (f, A)—or merely A when f is clear from the context—is *admissible* if $F(f, A)$ is open and closed in the set of all fixed points of f.

An open set with no fixed points on its boundary, an isolated fixed point, any isolated component of $F(f)$ is admissible. The class of admissible sets of a given map forms a Boolean algebra under the usual operations. The fixed point index L may easily be extended to all admissible pairs of X. In fact if (f, A) is admissible and U and V are neighborhoods of A whose closures are disjoint from $F(f, X - A)$, then the symmetric difference of U and V is free of fixed points, so $L(f, U) = L(f, V)$. Thus we may define $L(f, A)$ to be the common value of $L(f, U)$ for such U. With minor changes, properties L1245 and 3.1245 continue to hold for L a function on admissible pairs. Also

(3.6) *If $L(f, A) \neq 0$ and each component of $F(f, A)$ is isolated, then at least one component of $F(f, A)$ is essential.*

(3.7) *If K is an isolated component of $F(f)$ and $L(f, K) \neq 0$, then K is essential.*

(3.8) *If f has a finite number of fixed point and $L(f) \neq 0$, then at least one fixed point is essential.*

Briefly, when the components of $F(f)$ are isolated, a fixed point index suffices to locate and recognize essential components of $F(f)$.

4. Non-isolated components. When the components of $F(f)$ are not isolated—and hence not admissible—we show that a fixed point index still suffices to locate and recognize essential components, provided limit operations are introduced.

Let B be a Boolean algebra. We use the notation of [4] where, in particular, $\leq$ denotes the order relation on B, 0 the zero element of B, b' the complement of an element b of B. A non-empty subset $\mathfrak{S}$ of B is called a *system* provided $0 \notin \mathfrak{S}$ and for each pair $a, b \in \mathfrak{S}$ there is an element $c \in \mathfrak{S}$ such that $c \leq a \cap b$. The set of systems in B is partially ordered by defining $\mathfrak{S}'$ to be finer than $\mathfrak{S}$ (written $\mathfrak{S}' \geq \mathfrak{S}$) provided that if $a \in \mathfrak{S}$ there exists an element b of $\mathfrak{S}'$ such that $b \leq a$.

LEMMA 4.1. *Let B be a Boolean algebra and let S be a subset of B containing a non-zero element and such that if $a \cup b \in S$ and $a \cap b = 0$, then either a or b is in S. Then S contains a system maximal in B.*

Proof. S contains a (trivial) system and any collection of systems contains a maximal system. Thus we need only show that any system in S, maximal in S, is also maximal with respect to all systems in B.

So suppose $\mathfrak{S}$ and $\mathfrak{S}^*$ are systems in B and S respectively, with $\mathfrak{S} \geq \mathfrak{S}^*$ and $\mathfrak{S} \not\leq \mathfrak{S}^*$. We must show that $\mathfrak{S}^*$ is not maximal in S. That $\mathfrak{S} \not\leq \mathfrak{S}^*$ means there is an element $c \in \mathfrak{S}$ such that $s \not\leq c$ for all $s \in \mathfrak{S}^*$. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the collection of sets $s \cap c'$ and $s \cap c$, respectively, for all $s \in \mathfrak{S}^*$. Both $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are systems finer than $\mathfrak{S}^*$. For either $\mathfrak{S}_1$ or $\mathfrak{S}_2$, say $\mathfrak{S}_1$, we have $\mathfrak{S}_1 \cap S \geq \mathfrak{S}_1$. If $\mathfrak{S}^1$ denotes $\mathfrak{S}_1 \cap S$, then $\mathfrak{S}^1$ is easily seen to be a system. But $\mathfrak{S}^1 \subset S$ and $\mathfrak{S}^1 \geq \mathfrak{S}^*$. Also $\mathfrak{S}^* \not\geq \mathfrak{S}^1$, since $s \cap c \neq 0$ for all $s \in \mathfrak{S}^*$. Similar results hold if $\mathfrak{S}_2 \cap S \geq \mathfrak{S}_2$. Thus $\mathfrak{S}^*$ is not maximal in S and the proof is complete.

LEMMA 4.2. *If $f \in X^X$ let B be the Boolean algebra of admissible subsets of X. Let $\{A_i\}$ be a maximal system in B such that $\bigcup A_i$ is compact and let $D = \bigcap A_i$. Then D is one component of $F(f)$ if $F(f, A_i) \neq \phi$ for each i, otherwise $D = \phi$. D is essential if and only if each A_i is essential.*

Proof. The first assertion follows from these readily established facts: 1) $F(f, D)$ is a union of components of $F(f)$, since $\{A_i\}$ is maximal. 2) $F(f, D)$ is connected, since each A_i is admissible. 3) D contains a fixed

point if each A_i contains a fixed point, since then $\{F(f, A_i)\}$ is a system of compact sets.

It remains to show that if each A_i is essential, then D is essential. Let V be a neighborhood of D. Since $\{\bar{A}_i\}$ is a system of compact sets V contains an element of $\{A_i\}$ so that V is essential. Since V is open, maps near f have fixed points in V itself, consequently D is essential.

THEOREM 4.3. *Let $f \in X^X$ and let A be an admissible set whose closure is compact. If $L(f, A) \neq 0$, then A contains an essential component of $F(f)$.*

Proof. Choose B as in 4.2 and let S be the set of admissible subsets A' of A for which $L(f, A') \neq 0$. The set S contains a non empty element, so 4.1 applies, and 4.2 completes the proof.

COROLLARY 4.4. *If $L(f, A) \neq 0$ and $F(f, A)$ is totally disconnected, then f has an essential fixed point in A.*

For example, let f be the self-map of the unit interval for which $f(x) = x \sin^2 1/x$ if $x \neq 0$ and $f(0) = 0$. Then there is an essential point—which must be zero.

In order to recognize whether a given component K of $F(f)$ is essential, first note that K may always be expressed as the intersection of a maximal system Σ of admissible sets. Then K is essential if Σ can be chosen so that for each $A \in \Sigma$ we have $L(f, A) \neq 0$.

5. The Euclidean case. Let X be a finite polyhedron, L the trace function. In this section we consider admissible pairs (f, A), where A has a neighborhood G homeomorphic under a map h with an open subset of Euclidean n-space R^n. This Euclidean case is simpler chiefly because the notion of displacement map is applicable. If $f \in X^X$ and A is a subset of X such that $A \cup f(A) \subset G$, then the displacement map $d: A \to R^n$ is defined by $d(x) = h(f(x)) - h(x)$. Thus the study of fixed points of f is replaced by the study of zeroes of d.

Fix following notation: P is an orientable relative n-pseudomanifold [5] with boundary B consisting of the orientable $(n-1)$-pseudomanifolds $B_1, \cdots, B_k$. If $P \subset G$ and (f, P) is admissible, then the displacement map d of f maps B into $R^n - (0)$. In fact the Euclidean distance from $d(B)$ to $0 \in R^n$ is positive, so we may restrict ourselves to triangulations of R^n such that the origin is in an (open) n-simplex, always denoted by τ, which is disjoint from $d(B)$. We want to use the notion of degree of a map, hence we must specify orientations for the spaces involved. An orientation

of a space—or space modulo a subspace—is a generator of the appropriate cohomology group. We shall continue to use simplicial cohomology groups, but now with the integers as coefficients. The groups $H^n(P, B)$, $H^{n-1}(B_i)$, $H^n(R^n, R^n - \tau)$, $H^{n-1}(R^n - \tau)$, and $H^{n-1}(R^n - (0))$ are infinite cyclic. If we consider $H^{n-1}(B_i)$ as a subgroup of $H^{n-1}(B)$, the coboundary homomorphism δ^* of the pair (P, B) provides an isomorphism of $H^{n-1}(B_i)$ onto $H^n(P, B)$. We may suppose that the homeomorphism h is such that $h^{-1}(\tau) \subset P - B$, hence h induces an isomorphism

$$h^*: H^n(R^n, R^n - \tau) \to H^n(P, B).$$

The coboundary homomorphism δ^* of $(R^n, R^n - \tau)$ and the homomorphism induced by the inclusion map of $R^n - \tau$ into $R^n - (0)$ are also isomorphisms. Thus the natural orientation of the simplex $\tau \subset R^n$ determines a generator of $H^n(R^n, R^n - \tau)$ and thereby, via the isomorphisms mentioned above, orientations of $R^n - \tau$, B_i, $R^n - (0)$, and (P, B). In particular the degree q_i of d on B_i, that is, the degree of the map $d \mid B_i: B_i \to R^n - (0)$, is defined in terms of these orientations.

THEOREM 5.1. *Let the orientable relative n-pseudomanifold P be a subpolyhedron of X, and let the boundary B of P be the union of the orientable $(n-1)$-pseudomanifolds $B_1, \cdots, B_k$. If $f \in X^X$ has no fixed points on B and if $P \cup f(P) \subset G$, an n-dimensional Euclidean open set of X, then $L(f, P) = (-1)^n \sum_1^k q_i$, where q_i is the degree of d on B_i.*

Proof. Let d^* be the homomorphism of simplicial cohomology groups induced by the map $d: B \to R^n - \tau$. If ω_i and θ' denote the orientations assigned to B_i and $R^n - \tau$, then $d^*(\theta') = \Sigma q_i \omega_i$. But $\delta^*(\omega_i) = \omega$, the selected orientation of (P, B), hence $\delta^*(d^*(\theta')) = (\Sigma q_i)\omega$. Also $\delta^*(\theta') = \theta$, the selected orientation of $(R^n, R^n - \tau)$. Thus $d^*(\delta^*(\theta')) = q\omega$, where q is the degree of the map $d: (P, B) \to (R^n, R^n - \tau)$. The following diagram is commutative:

$$\begin{array}{ccc} H^n(R^n, R^n - \tau) & \xrightarrow{d^*} & H^n(P, B) \\ \uparrow \delta^* & & \uparrow \delta^* \\ H^{n-1}(R^n - \tau) & \xrightarrow{d^*} & H^{n-1}(B) \end{array}$$

where, in each case, δ^* is the coboundary homomorphism and d^* the homomorphism induced by d. Consequently $q = \Sigma q_i$. It is known (par. 3, § 3, [6]) that the map f has a simplicial approximation f' arbitrarily near f and such that only n-simplexes contain fixed points. Because of L3 we may

assume that f has this property. Hence $L(f, P) = (-1)^n \Sigma(s_j, f^+(s_j))$, where the sum extends only over simplexes which contain fixed points. If necessary we alter the positive orientation of each such simplex σ_j to agree with the selected orientation of (P, B). This does not change $L(f, P)$. Now choose a simplicial approximation d' of the displacement map d of f such that $d'(\sigma) = \tau$, the n-simplex containing the origin of R^n, if and only if σ contains a fixed point of f. Using the fact that $hf \mid Bd\sigma_j$ and $d' \mid Bd\sigma_j$ are homotopic in $R^n - (0)$ for each simplex σ_j containing a fixed point, it is easily verified that $(s_j, f^+(s_j)) = \pm 1$ depending on whether $d' : \sigma_j \to \tau$ preserves or reverses orientation. But since the simplicial map $d' : (P, B) \to (R^n, R^n - \tau)$ induces d^*, we conclude that $\Sigma(s_j, f^+(s_j)) = q$, and the result follows.

THEOREM 5.2. *Let $f \in X^X$ and let K be a component of $F(f)$ isolated in a Euclidean neighborhood G of dimension $n > 1$. Let $r_1, \cdots, r_k$ be integers such that $\Sigma r_i = L(f, K)$ and let $x_1, \cdots, x_k$ be distinct points of K. Then there is a map $f' \in X^X$ arbitrarily near f such that $x_1, \cdots, x_k$ are the only fixed points of f' in G and $L(f', x_i) = r_i$.*

Proof. Let S be a solid sphere centered at the origin of R^n. Triangulate G so that if $P = \mathrm{Cl\,St}\, K$, then $P \subset d^{-1}(S)$, where d is the displacement map of f. The polyhedron P is an orientable relative n-pseudomanifold. By further subdividing P we may assume that, if B is the boundary of P and $V_i = \mathrm{St}(x_i)$, then the sets $B, \bar{V}_1, \cdots, \bar{V}_k$ are mutually disjoint. It follows that $Q = P - \bigcup V_i$ is also a relative n-pseudomanifold. Let ω be the orientation of Q selected as in the previous theorem. If C_i is the boundary of V_i, then the boundary C of Q is $B \cup \bigcup C_i$.

Now let $e_i : \bar{V}_i \to S$ be a map such that x_i is the only zero of e_i in $\bar{V}_i$ and—measured in terms of the orientation on C_i induced by that of Q—$e_i \mid C_i$ has degree $-(-1)^n r_i$. The maps $e_i \mid C_i$ and $d \mid B$ constitute a map $d_1 : C \to S - (0)$. Let θ be the orientation of $S - (0)$ corresponding to that of $R^n - (0)$. It follows from the previous theorem and our definition of d_1 that $\delta^*(d_1^*(\theta)) = (L(f, K) - \Sigma r_i)\omega = 0$. By the exactness of the cohomology sequence of (Q, C) we conclude that $d_1^*(\theta)$ is contained in the image of $H^{n-1}(Q)$ in $H^n(C)$ under the homomorphism induced by the inclusion map of C into Q.

Since $S - (0)$ is a homotopy $(n-1)$-sphere a generalization of the Hopf Extension theorem [7] applies and d_1 has an extension $d' : Q \to S - (0)$. Using the maps e_i and d_1 we obtain an extension of $f \mid X - P$ to a map $f' \in X^X$ with $x_1, \cdots, x_k$ as its only fixed points in G. Furthermore when S is sufficiently small, f' is arbitrarily near f. Finally we note that $L(f, x_i)$ is the

degree of e_i on C_i measured in terms of the orientation of C_i induced by V_i—not Q—so $L(f, x_i) = +r_i$.

The form of the theorem when $n = 1$ is obvious.

Two consequences of the theorem are noteworthy. When $L(f, K) = 0$ we may take the set $\{x_1, \cdots, x_k\}$ to be empty, so K is inessential.

COROLLARY 5.3. *If K is a component of $F(f)$ isolated in a Euclidean neighborhood, then K is essential if and only if $L(f, K) \neq 0$.*

Next we see that such components are *minimal* among all essential sets.

COROLLARY 5.4. *If K is a component of $F(f)$ isolated in a Euclidean neighborhood and containing more than one point, then no proper subset of K is essential.*

6. An example. For a wide class of spaces the essentiality of a subset with respect to a map depends only on the behavior of the map in a neighborhood of the set.

LEMMA 6.1. *Let A be a subset of a polyhedron X. If the maps $f, g \in X^X$ agree on a compact neighborhood of A, then A is essential with respect to g if and only if essential with respect to f.*

Proof. Assume that A is inessential with respect to f and that f and g agree on a compact neighborhood U of A. Let $V \subset U$ be a neighborhood of A such that maps f' arbitrarily near f have no fixed points in V, and let $F \subset V$ be a compact neighborhood of A. If N is a neighborhood of g we will show that N contains a map g' with no fixed points in F. Consider for each f' (as above) the partial map g_1 which agrees with f' on F and g on $X - V$. Given a neighborhood P of $f \mid F \in X^F$ we may choose f' so near f that $f' \mid F$ and $f \mid F$ can be joined by a homotopy h_t such that for each t we have $h_t \in P$. Thus a similar homotopy joins g_1 to $g \mid F \cup (X - V)$. Since X is an ANR it follows from Borsuk's Theorem [3] that g has an extension to $g' \in X^X$. Furthermore if P is sufficiently small, g' is arbitrarily near g (always in the compact-open topology). Since it has no fixed points in F, g' is the required map.

The preceding lemma lets us use the trace function on infinite polyhedra. As an example we investigate the essentiality of sets with respect to the following self-map of the Euclidean space R^4. Let R^3 be the subspace of points (x_1, x_2, x_3, x_4) for which $x_4 = 0$. Let S^2 be the unit sphere in R^3. Let $P(y)$ be the plane determined by $y \in S^2$ and the x_4-axis. It is easy to see

there is a self-map f_y of $P(y)$ such that: a) Near y (and $-y$) points move directly away from y (and $-y$). b) In a neighborhood of the origin, f_y is a "flow" whose streamlines are rectangular hyperbolas with one asymptote the line through y and $-y$. Near this line the flow is toward the origin. c) Only $y, -y$, and the origin are fixed points. If f_y is defined congruently on every plane $P(y)$, $y \in S^2$ we obtain a self-map f of R^4. $F(f)$ consists of the origin 0, and a 2-sphere Σ. Let A be a compact neighborhood of $F(f)$. Extend $f: A \to R^4 \subset S^4$ to a self-map F of S^4. Using 6.1 we find that any subset interior to A which is essential with respect to F in S^4 is essential with respect to f in R^4.

Now we calculate $L(F, 0)$ and $L(F, \Sigma)$. Choose a neighborhood of Σ whose boundary B is topologically the product of a circle and a 2-sphere. The displacement map d on B has the same degree as the spherical representation $r: B \to S^3$ hence $L(F, \Sigma) = 2$. On the boundary S of a small solid sphere centered at the origin, d has the same degree as the antipodal map of an equitorial 2-sphere of S, hence $L(F, 0) = -1$.

Thus both components of $F(f)$ are essential and by 5.4 also minimal. In fact the essential subsets of R^4 are exactly those which contain either the origin or the fixed sphere Σ.

7. Uniqueness of the trace function. We prove that the only topologically invariant fixed point index on the class of finite polyhedra is the trace function L. Let X be a finite polyhedron. The main fact needed is the following slight generalization of a known theorem (par. 3, § 3, [6]).

THEOREM 7.1. *If $f \in X^X$ and T is a triangulation of X there is a simplicial approximation f' of f from T' to T with the property that if $\sigma \in T'$ and $\sigma \subset f(\sigma)$, then σ is not a face of another simplex of T'.*

The proof is omitted.

A map $f \in X^X$ is said to be *normal* provided each of its fixed points x has an n-cell neighborhood K with boundary S (n depending on x) such that K contains no other points of $f^{-1}(x) \cup F(f)$, and $f(S)$ contains no interior points of K. For such a point $L(f, x) = (-1)^n q$, where q is the degree of f at x. The map f' in the preceding theorem is normal, for if x is a fixed point of f' lying in the interior of the simplex σ, then $\sigma \subset f'(\sigma)$. Consequently $F(f', Bd\sigma)$ is empty, and σ is an open set of X. Thus $L(f, U) = \Sigma(-1)^{n_i} q_i$, where the summation is over the fixed points of any sufficiently close normal approximation of f. We have proved

COROLLARY 7.2. *If U is an open set of X with no fixed points of $f \in X^X$ on its boundary, then $L(f, U)$ is a topological invariant of (f, U), that is,* L5 *holds with the map h only required to be a homeomorphism.*

We now show that L is unique.

THEOREM 7.3. *Let $\mathfrak{S}$ be a set of finite polyhedra including an n-cell and an n-sphere for each $n > 0$. If M is a topologically invariant fixed point index on each $X \in \mathfrak{S}$, then $M = L$, the trace function.*

Proof. If $f \in X^X$, $U \subset X$, $X \in \mathfrak{S}$, and the boundary of U is free of fixed points we must show $M(f, U) = L(f, U)$. We claim that it suffices to calculate $M(f, \sigma)$ in this special situation:

1) σ is an n-simplex of X with $F(f, Bd\sigma) = \phi$

2) $\bar{\sigma} \cup f(\bar{\sigma}) \subset G$, an n-dimensional Euclidean open set

3) the degree b of d on the boundary S of σ is ± 1.

By the previous discussion any map has an arbitrary close approximation f each of whose fixed points is contained in such a σ. By Theorem 5.1, $L(f, \sigma) = (-1)^n b$. If the same is true for M, it will follow from L2 and L3 that M and L are identical.

To prove that $M(f, \sigma) = (-1)^n b$ we first show that for this σ, $M(f, \sigma)$ depends only on the degree b. Then we calculate M for standard maps of degree $+1$ and -1. Let f_1 be another map for which 1) and 2) above hold and whose displacement map d_1 also has degree b on S. Since $d \mid S$ and $d_1 \mid S$ have the same degree they can be joined by a homotopy $D: S \times I \to R^n - (0)$. Let V be a neighborhood of $\bar{\sigma}$ such that $V \cup f(V) \subset G$. We now define a homotopy $H: X \times I \to X$ of f.

H is defined on a closed subset of $X \times I$ by $H(x, t) = h^{-1}(h(x) + D(x, t))$ if $x \in S$ and $H(x, t) = f(x)$ if $x \notin V$ or $t = 0$, finally $H(x, t) = f_1(x)$ if $x \in \sigma$ and $t = 1$. We may assume that G is homeomorphic to R^n, thus there is an extension of this partial map to a homotopy $H: X \times I \to X$ from f to the map f' such that $f'(x) = H(x, 1)$. No fixed points appear on S during the homotopy, consequently L3 implies that $M(f, \sigma) = M(f', \sigma)$. But $f' \mid \bar{\sigma} = f_1 \mid \bar{\sigma}$ so we have $M(f, \sigma) = M(f_1, \sigma)$.

It remains to verify $M(f, \sigma) = (-1)^n b$ for particular maps f with $b = \pm 1$. For $b = 1$, choose h so that $h(S)$ is the unit sphere centered at the origin of R^n, then choose f so that on σ, $d = f$. Now imbed $\bar{\sigma}$ in an n-sphere S^n as its northern hemisphere. There is an obvious extension

of f to a self-map F of S^n such that F is homotopic to the identity map and F is a self-map of the southern hemisphere H of S^n. Consequently,

$$M(f, \sigma) = M(F, \sigma) = L(F) - L(F \mid H) = 1 + (-1)^n - 1 = (-1)^n.$$

If $b = -1$ and n is odd, we may suppose $f(\sigma) \subset \sigma$, so

$$M(f, \sigma) = 1 = (-1)^n b.$$

If $b = -1$ and n is even, the proof is similar to the case $b = 1$, with F a self-map of an n-cell with two fixed points outside σ.

Note that the proof does not use L1.

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SYMMETRIZATION RESULTS FOR SOME CONFORMAL INVARIANTS.*

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1. In recent years considerable attention has been paid to symmetry results and methods in the theory of functions and potential theory. A good deal of this development is due to Pólya and Szegö [7]. In particular such results have been proved for the characteristic conformal invariants of quadrangles and doubly-connected domains in various situations and for various types of symmetrization. As these invariants can be expressed in terms of the extremal lengths of certain curve families associated with the domains it is natural to expect that corresponding results hold for generalizations of extremal length. We prefer to work with quantities which we may call modules and which are essentially the reciprocals of extremal lengths. These modules are either numbers or functions associated in a conformally invariant manner with certain configurations. We will not attempt here an exhaustive enumeration of all symmetry results of this type but will give several examples clarifying the method. Also we will give an interesting application to the theory of univalent functions. We will work exclusively with circular symmetrization.

For purposes of reference we will collect here the definitions of the modules with which we will have to do, together with various explanatory remarks. In order to avoid repetition we make the following conventions throughout. In each case we deal with a domain D of finite connectivity lying in the $z(=x+iy)$ plane. On D we consider the class of functions $\rho(x, y)$ which are non-negative real valued functions of integrable square over D and subject to certain auxiliary conditions to be stated explicitly in each case. On any boundary component of D there is a natural cyclic order among the prime ends and a natural sense of description. In particular we may speak of a finite number of prime ends dividing a boundary component into sides. With D also will be associated certain classes of curves. These may be Jordan curves or open arcs on D tending to a prime end on the boundary of D at either extremity. In any case we will assume them to be locally rectifiable, i. e., if a closed subarc lies in the neighborhood associated with a local uniformizing

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parameter, its image in the plane of the local uniformizing parameter shall be rectifiable. This definition is conformally invariant and can be applied even in the case of Riemann surfaces. In each case the module is given by the minimum of $\int\int_D \rho^2\, dxdy$ as ρ runs over the class of functions indicated above. The minimum is attained in all cases with which we deal here, and the corresponding metric $\rho\,|\,dz\,|$ will be called the extremal metric. The extremal metric is always unique, apart from trivial modifications.

I. D a quadrangle (i. e., a simply-connected domain with four prime ends distingiushed on its boundary, numbered 1, 2, 3, 4 in the natural sense). Γ the class of open arcs joining the sides 12 and 34. The auxiliary condition is that, for $\gamma \in \Gamma$, $\int_\gamma \rho\,|\,dz\,|$ exists (possibly having the value $+\infty$, this will be tacitly assumed in the later cases) with $\int_\gamma \rho\,|\,dz\,| \geqq 1$. The corresponding minimum will be denoted by M and called the module of D. It is a characteristic conformal invariant of D. If the quadrangle is mapped on a rectangle R with 1, 2, 3, 4 going into the vertices A_1, A_2, A_3, A_4 of the rectangle and A_2A_3 has length 1, then A_1A_2 has length M [2], pp. 328-329.

II. D a doubly-connected domain. C the class of Jordan curves separating its boundary continua $\mathfrak{K}_1$, $\mathfrak{K}_2$ neither of which shall reduce to a point The auxiliary condition is that, for $c \in C$, $\int_c \rho\,|\,dz\,|$ exists and $\geqq 1$. The corresponding minimum will be denoted by $\mathfrak{M}$ and called the module of D. It is a characteristic conformal invariant of D. If D is mapped on the circular ring $r_1 < |\,w\,| < r_2$, then $\mathfrak{M} = (1/2\pi)\log(r_2/r_1)$.

III. D a pentagon (i. e., a simply-connected domain with five prime ends distinguished on its boundary, numbered 1, 2, 3, 4, 5 in the natural sense.) $\Gamma_1(\Gamma_2)$ the class of open arcs joining the side 12 to 34(45). The auxiliary conditions are that, for $\gamma_i \in \Gamma_i$, $\int_{\gamma_i} \rho\,|\,dz\,|$ exists and that

$$\int_{\gamma_1} \rho\,|\,dz\,| \geqq a_1, \qquad \int_{\gamma_2} \rho\,|\,dz\,| \geqq a_2,$$

where a_1, a_2 are certain two non-negative numbers not both zero. The corresponding minimum will be denoted by $M(a_1, a_2)$. It is a function of a_1, a_2 called the module of D. Its existence is proved in [2], pp. 330-333.

IV. D a doubly-connected domain with one distinguished interior point P, neither of its boundary continua $\mathfrak{K}_1$, $\mathfrak{K}_2$ reducing to a point. Let C_1 denote the class of Jordan curves lying in D and separating $\mathfrak{K}_1$ from $\mathfrak{K}_2$ and P, let C_2 denote the class of Jordan curves lying in D and separating $\mathfrak{K}_2$ from $\mathfrak{K}_1$ and P. The auxiliary conditions are that, for $c_i \in C_i$, $(i = 1, 2)$, $\int_{c_i} \rho \mid dz \mid$ exists and that $\int_{c_1} \rho \mid dz \mid \geqq a_1$, $\int_{c_2} \rho \mid dz \mid \geqq a_2$, where a_1, a_2 are certain two non-negative numbers not both zero. The corresponding minimum will be denoted by $\mathfrak{M}(a_1, a_2)$. It is a function of a_1, a_2, called the module of D. The existence of this quantity is not given explicitly in [2] but follows from the pentagon problem in the same way that the existence of a module for a triply-connected domain follows from the hexagon problem [2], p. 348.

V. D a triply-connected domain none of whose boundary continua $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$ reduces to a point. Let $C_1(C_2, C_3)$ denote the class of Jordan curves lying in D and separating $\mathfrak{K}_1(\mathfrak{K}_2, \mathfrak{K}_3)$ from $\mathfrak{K}_2$ and $\mathfrak{K}_3(\mathfrak{K}_3$ and $\mathfrak{K}_1$, $\mathfrak{K}_1$ and $\mathfrak{K}_2)$. The auxiliary conditions are that, for $c_i \in C_i$ $(i = 1, 2, 3)$, $\int_{c_i} \rho \mid dz \mid$ exists and

$$\int_{c_1} \rho \mid dz \mid \geqq a_1, \qquad \int_{c_2} \rho \mid dz \mid \geqq a_2, \qquad \int_{c_3} \rho \mid dz \mid \geqq a_3,$$

where a_1, a_2, a_3 are certain three non-negative numbers not all zero. The corresponding minimum will be denoted by $\mathfrak{M}(a_1, a_2, a_3)$ and called the module of D. It is a function of a_1, a_2, a_3. Its existence for all values of a_1, a_2, a_3 is proved in [2], pp. 336-342.

2. LEMMA 1. *Let D be a doubly-connected domain in the z-plane $(z = x + iy)$ with boundary continua $\mathfrak{K}_1$ and $\mathfrak{K}_2$. Let $\omega(x, y)$ be the harmonic measure of $\mathfrak{K}_1$ relative to D. Let $D(\omega)$ denote the Dirichlet integral of ω over D. Then the module of D is equal $1/D(\omega)$.*

Since both the module and the Dirichlet integral are conformally invariant we may suppose D to be the circular ring $r_1 < \mid z \mid < r_2$ ($\mathfrak{K}_1$ being $\mid z \mid = r_2$). In this case $\omega = (\log \mid z \mid / r_1)/(\log r_2/r_1)$. Direct calculation gives $D(\omega) = 2\pi/(\log r_2/r_1)$, i. e., the reciprocal of the module.

LEMMA 2. *Let D be a quadrangle such that the sides are piecewise smooth curves, lying in the z-plane $(z = x + iy)$. Let $u(x, y)$ be a function harmonic on D, continuous on $\bar{D}$, equal to 0 on the side 12, equal to 1 on the*

side 34 *and with* $\partial u/\partial n = 0$ *on* 23 *and* 41. *Let* $D(u)$ *denote the Dirichlet integral of* u *over* D. *Then the module of* D *is equal* $1/D(u)$.

As before we can assume D to be the rectangle $0 < x < l$, $0 < y < 1$ with vertices $(0, 0)$, $(l, 0)$, $(l, 1)$, $(0, 1)$. The function in question is then $u(x, y) \equiv y$ and direct calculation gives the result.

LEMMA 3. *Let* D *be a triply-connected domain in the* $z(= x + iy)$ *plane with boundary continua* $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$ *as in* **V**. *Let* D_i $(i = 1, 2, 3)$ *be doubly-connected domains lying in* D *and not overlapping such that* D_i *separates* $\mathfrak{K}_i$ *from the other two boundary continua of* D. *Let* $\mathfrak{M}_i$ *be the module of* D_i. *Then*

$$\mathfrak{M}(a_1, a_2, a_2) = \max(a_1^2\mathfrak{M}_1 + a_2^2\mathfrak{M}_2 + a_3^2\mathfrak{M}_3)$$

as D_1, D_2, D_3 *range over all sets of three doubly-connected domains with the above properties, it being understood that one or two of the domains* D_i *may be taken as degenerate, i. e., having module* 0.

Indeed let D_1, D_2, D_3 be such a trio of domains. Let ρ be the extremal function in the problem **V** for D and the values a_1, a_2, a_3. If any a_i vanishes, we may clearly leave it out of consideration. Otherwise in D_i, ρ/a_i is admissible (i. e., satisfies the auxiliary conditions) in the module problem for that domain. Thus $(1/a_1^2)\iint_{D_1} \rho^2\, dxdy \geqq \mathfrak{M}_1$ and

$$\mathfrak{M}(a_1, a_2, a_3) = \iint_D \rho^2\, dxdy \geqq a_1^2\mathfrak{M}_1 + a_2^2\mathfrak{M}_2 + a_3^2\mathfrak{M}_3. \tag{1}$$

On the other hand, a triply-connected domain D is divided by its line of symmetry into two hexagons. From the canonical domains of [2], Fig. 3, p. 337 associated with these for various values of a_1, a_2, a_3 we can obtain representations of D by suitable identifications (see [4], p. 148). The domains so obtained are naturally divided into three doubly-connected domains for which equality is attained in (1). In the degenerate cases one or two of these doubly-connected domains will be missing.

Remark. Lemma 3 is only a very special case of an important general principle to which we will probably return in a later paper.

Let now D be a triply-connected domain for which the boundary $\mathfrak{K}_3$ is a circle $|z| = R$ and $\mathfrak{K}_1$ and $\mathfrak{K}_2$ lie interior to this circle. With D we associate a domain $\boldsymbol{D}$ obtained by circular symmetrization in the following manner. For r, $0 < r < R$, let $l_1(r)$, $l_2(r)$ be the angular Lebesgue measures of the

intersections of $\mathfrak{K}_1$ and $\mathfrak{K}_2$ with $|z| = r$. We define $\mathfrak{E}_1$ as the boundary of the set: $-\frac{1}{2}l_1(r) \leqq \phi \leqq \frac{1}{2}l_1(r)$ where r, ϕ are polar coordinates and $\mathfrak{E}_2$ as the boundary of the set; $\pi - \frac{1}{2}l_2(r) \leqq \phi \leqq \pi + \frac{1}{2}l_2(r)$. That these are closed sets follows by semi-continuity considerations [9], p. 598 or by approximating $\mathfrak{K}_1$, $\mathfrak{K}_2$ by analytic curves, performing the above operation and passing to the limit. Evidently $\mathfrak{E}_1$ and $\mathfrak{E}_2$ do not intersect one another or $|z| = R$. Then $\boldsymbol{D}$ is the domain bounded by $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{K}_3$ ($= \mathfrak{E}_3$, say). Let the module of $\boldsymbol{D}$ be denoted by $\mathfrak{M}(a_1, a_2, a_3)$.

THEOREM 1. $\mathfrak{M}(a_1, a_2, 0) \geqq \mathfrak{M}(a_1, a_2, 0)$.

First let us assume $\mathfrak{K}_1$ and $\mathfrak{K}_2$ to be analytic curves. For either one of the hexagons into which D is divided by its line of symmetry the canonical domain associated with the values $\frac{1}{2}a_1$, $\frac{1}{2}a_2$, 0 will be of type 2, 4 or 5 [2], pp. 337, 338. The corresponding canonical domain for D is obtained by taking two of these, one obtained from the other by reflection, and identifying appropriate sides. Assuming the canonical domain to be situated so that the images of $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$ are parallel to the imaginary axis, the real part of the function mapping D on this domain is a singled-valued function u defined on D. (The imaginary part of the mapping function will not be single-valued). The function u may be assumed to have the value 0 on $\mathfrak{K}_3$, a constant value $\beta_1 \geqq 0$ on $\mathfrak{K}_1$ and a constant value $-\beta_2 \leqq 0$ on $\mathfrak{K}_2$. The curves Θ on which u vanishes divide D into two doubly-connected domains D_1 and D_2. D_i has $\mathfrak{K}_i$ $(i = 1, 2)$ as one boundary, the other being made up from Θ and $\mathfrak{K}_3$. The function u is harmonic in each of D_1 and D_2 although not necessarily in D. Moreover $\mathfrak{M}(a_1, a_2, 0) = a_1^2(D(u/\beta_1; D_1))^{-1} + a_2^2(D(u/\beta_2; D_2))^{-1}$ (Lemmas 1 and 3, the second variable denotes the domain relative to which we are taking the Dirichlet integral.).

Consider the surface $u = u(x, y)$ over D completed so that $u \equiv \beta_1$ on the set bounded by $\mathfrak{K}_1$, and $u \equiv -\beta_2$ on the set bounded by $\mathfrak{K}_2$. We apply to it circular symmetrization with respect to the half-plane through the positive x-axis and perpendicular to the (x, y)-plane. In this way we obtain a function $\tilde{u}(x, y)$, still defined over $|z| \leqq R$. The set on which $\tilde{u} \equiv \beta_1$ is precisely that bounded by $\mathfrak{E}_1$, the set on which $\tilde{u} \equiv -\beta_2$ is precisely that bounded by $\mathfrak{E}_2$. The set on which $\tilde{u}$ vanishes divides $\boldsymbol{D}$ into two doubly-connected domains $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ corresponding to D_1 and D_2. A familiar argument [7], pp. 185, 194 shows that $D(\tilde{u}; \boldsymbol{D}_i) \leqq D(u; D_i)$, $(i = 1, 2)$. If, then, $\tilde{\omega}_i$ is the harmonic measure with respect to $\boldsymbol{D}_i$ of $\mathfrak{E}_i$ we have, by Dirichlet's principle, $D(\tilde{\omega}_i; \boldsymbol{D}_i) \leqq D(\tilde{u}/\beta_i; \boldsymbol{D}_i) \leqq D(u/\beta_i; D_i)$. By Lemma 3,

$$\mathfrak{M}(a_1, a_2, 0) \geqq a_1^2 (D(\tilde{\omega}_1; \boldsymbol{D}_1))^{-1} + a_2^2 (D(\tilde{\omega}_2; \boldsymbol{D}_2))^{-1}$$
$$\geqq a_1^2 (D(u/\beta_1; D_1))^{-1} + a_2^2 (D(u/\beta_2; D_2))^{-1}.$$

Thus $\mathfrak{M}(a_1, a_2, 0) \geqq \mathfrak{M}(a_1, a_2, 0)$ as stated.

To remove the restriction that $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are analytic curves we observe that any continua can be approximated from outside by such curves and the modules in question are readily seen to behave continuously under this approximation.

Let now D be a doubly-connected domain in the z-plane with the point at infinity as a distinguished interior point. Then its boundary continua $\mathfrak{K}_1$ and $\mathfrak{K}_2$ lie in a bounded part of the plane. Let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ correspond to $\mathfrak{K}_1$ and $\mathfrak{K}_2$ under symmetrization as above. Let $\boldsymbol{D}$ be the doubly-connected domain bounded by $\mathfrak{K}_1$ and $\mathfrak{K}_2$, the point at infinity again being a distinguished interior point. Let $\mathfrak{M}(a_1, a_2)$ and $\mathfrak{M}(a_1, a_2)$ be respectively the modules of D and $\boldsymbol{D}$ as in **IV**.

THEOREM 2. $\mathfrak{M}(a_1, a_2) \geqq \mathfrak{M}(a_1, a_2)$.

This result is easily derived from Theorem 1 by regarding D and $\boldsymbol{D}$ as the limits of their intersections with a circle centered at the origin. The latter are triply-connected domains for which Theorem 1 applies and the appropriate modules tend to $\mathfrak{M}(a_1, a_2)$ and $\mathfrak{M}(a_1, a_2)$ as the radius of the above circle tends to infinity. The result can also be obtained directly.

3. Consider now the unit circle $|z| < 1$ and the points $-r_1, 0, r_2, 1, -1$ $(0 < r_1, r_2 < 1)$ on its diameter along the real axis. Let the upper semicircle be mapped on the upper half Z-plane with these points going into Z_1, Z_2, Z_3, Z_4, Z_5 (in ascending order of magnitude on the real axis). Consider the function

$$\zeta = K \int_{Z_1}^{Z} [(Z - Z^*)^2 / \{(Z - Z_1)(Z - Z_2)^2 (Z - Z_3)(Z - Z_4)(Z - Z_5)\}]^{\frac{1}{2}} dZ.$$

where K is a constant and Z^* is a real value in the interval $Z_4 \leqq Z^* \leqq Z_5$. For suitable choice of K this maps the upper half Z-plane on a domain bounded by rectilinear segments of the following nature. Let $A_1, A_2, A_3, A_4, A^*, A_5$ be the images of $Z_1, Z_2, Z_3, Z_4, Z^*, Z_5$. Then A_1A_2 is a half-infinite horizontal segment, A_2 being at infinity in the direction of increasing $\Re\zeta$; A_2A_3 is a half-infinite horizontal segment, the value of $\Im\zeta$ on it being larger than on A_1A_2 and $\Re\zeta$ decreasing as we go from A_2 to A_3; A_3A_4 is a vertical segment, $\Im\zeta$ decreasing as we go from A_3 to A_4; A_4A^* is a horizontal segment,

$\Re\zeta$ increasing as we go from A_4 to A^*; A^*A_5 is a horizontal segment, $\Re\zeta$ decreasing as we go from A^* to A_5; A_5A_1 is a vertical segment $\Im\zeta$ decreasing as we go from A_5 to A_1. Exceptionally A^* may coincide with A_4 or A_5 when Z^* coincides with Z_4 or Z_5 and the corresponding segment reduce to a point.

Now let the domain bounded by the segments A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_1 (i. e., obtained from the preceding by removing the slit penetrating to the point A^*) be mapped on the upper half w-plane with A_1, A_2, A_3, A_4, A^*, A_5 going into w_1, w_2, w_3, w_4, w^*, w_5 where $w_2 = 0$ and w_4 or $w_5 = \infty$ according as $\Re\zeta$ is smaller at A_4 or A_5. Reflecting across the segments $-1 < z < 1$ and $w_5w_1w_4$ of the real axis in the z- and w-planes we obtain a mapping of $|z| < 1$ on the w-plane minus a forked slit. We call this domain E_w. If $w_4 = \infty$, the slit runs in along the negative real axis to w_5 where it forks along curves running to w^* and $\bar{w}^*$. If $w_5 = \infty$, it runs in along the positive real axis to w_4 where it forks in a similar manner. By suitable choice of the constant K we may assume that $dw/dz = 1$ at $z = 0$.

If we extend ζ as a (non-single-valued) function of w to the whole w-plane by reflection in the various segments w_1w_2, w_2w_3, w_3w_4, w_4w_5, w_5w_1, we see at once that $d\zeta^2$ is a quadratic differential on the w-sphere, positive on $w_1w_2w_3$ and w_4w_5 and also on the slits running from $w_5(w_4)$ to w^* and $\bar{w}^*$, negative on w_5w_1 and w_3w_4. Further it has simple poles at w_1, w_3, w_4 or w_5, a simple zero at w_5 or w_4 (respectively) and a double pole at w_2. This is easily verified by introducing appropriate local uniformizing parameters at these points (as in [6]). Let us denote $d\zeta^2 = Q(w)dw^2$. The curves on which $Q(w)dw^2 > 0$ will be called trajectories. The curves on which $Q(w)dw^2 < 0$, being the orthogonal trajectories of the preceding except at critical points, will be called orthogonal trajectories.

By the way in which $Q(w)dw^2$ was obtained it is clear that the orthogonal trajectories near w_2 are closed Jordan curves. This also follows from the general theory according to which, further, they approach circular form as they shrink down to w_2 [8]. Let one of these small orthogonal trajectories be denoted by L. Let the portion of the w-plane exterior to L and slit along the real axis from w_1 to w_3 be denoted by $\mathfrak{D}$. Let $\mathfrak{C}_1$ be the class of rectifiable arcs lying in $\mathfrak{D}$ and running from L back to L separating w_3 from infinity. Let $\mathfrak{C}_2$ be the class of rectifiable arcs lying in $\mathfrak{D}$ and running from L back to L separating w_1 from infinity.

If we continue the trajectories through w^* and $\bar{w}^*$ in the direction of description from $w_5(w_4)$ to these points they run into w_2 and separate all remaining trajectories on E_w into two sets. Except for the segments w_1w_2, w_2w_3 the portions of the trajectories in one set lying in $\mathfrak{D}$ belong to $\mathfrak{C}_1$; for

the other, they belong to $\mathfrak{C}_2$. Let these respective subsets of $\mathfrak{C}_1$, $\mathfrak{C}_2$ be denoted by $\mathfrak{C}_1^*$, $\mathfrak{C}_2^*$.

In the Q-metric, $|Q(w)|^{\frac{1}{2}}|dw|$, all curves of $\mathfrak{C}_1^*$ have the same length, say a_1, and all curves of $\mathfrak{C}_2^*$ have the same length, say a_2. If $w_4 = \infty$, $a_1 \geqq a_2$ and if $w_5 = \infty$, $a_1 \leqq a_2$, equality occurring when w_4, w_5 coincide at infinity. For definiteness we will, in the remainder of this section, suppose that the first case obtains, the second being quite analogous.

We now regard the following module problem: let ρ denote a non-negative real valued function, defined and of integrable square over $\mathfrak{D}$ and such that $\int_{c_i} \rho\,|dw|\,(c_i \in \mathfrak{C}_i, \quad i = 1, 2)$ exist with $\int_{c_1} \rho\,|dw| \geqq a_1$, $\int_{c_2} \rho\,|dw| \geqq a_2$, where a_1, a_2 are the non-negative numbers given above. It is required to determine ρ so that $\iint_{\mathfrak{D}} \rho^2\,dudv$ $(w = u + iv)$ is a minimum.

It is easily verified that $|Q(w)|^{\frac{1}{2}}$ is precisely the minimizing function. The corresponding minimum is denoted by $M^*(a_1, a_2)$ and called the module of the given configuration.

The curves of $\mathfrak{C}_1^*$ sweep out a quadrangle R_1 which has two opposite sides s_{11}, s_{13} composed of arcs of L, one further side s_{12} being the segment of the positive real axis from L to w_3 described twice, the last s_{14} being composed of the portions of the trajectories through w^* and $\bar{w}^*$ outside of L and the segment of the negative real axis from w_4 to w_5 described twice. Similarly the curves of $\mathfrak{C}_2^*$ sweep out a quadrangle R_2 which has two opposite sides s_{21} and s_{23} composed of arcs of L, one further side s_{22} being the segment of the negative real axis from w_1 to L described twice, the last s_{24} being composed of the portions of the trajectories through w^* and $\bar{w}^*$ outside of L. Let M_1 and M_2 be the modules of R_1 and R_2, in each case the class of curves Γ joining the two sides which lie on L.

As in the proof of Lemma 3 we see at once $M^*(a_1, a_2) = a_1^2 M_1 + a_2^2 M_2$.

4. Let S denote the class of functions $f(z)$, regular and univalent for $|z| < 1$ whose Taylor expansions about $z = 0$ begin

$$f(z) = z + b_2 z^2 + b_3 z^3 + \cdots + b_n z^n + \cdots.$$

It is well known that $r_1/(1 + r_1)^2 \leqq |f(-r_1)| \leqq r_1/(1 - r_1)^2$ for $0 < r_1 < 1$. In **3** we have constructed a class of functions $f_{Z^*}(z)$ contained in S and depending on the real parameter Z^*, these being the functions w carrying out the mapping indicated. As Z^* varies from Z_4 to Z_5 the function

$f_{Z^*}(z)$ varies from $z/(1+z)^2$ to $z/(1-z)^2$. It is easily seen that $f_{Z^*}(-r_1)$ varies continuously with Z^* and thus $|f_{Z^*}(-r_1)|$ takes each value in the range $(r_1/(1+r_1)^2, r_1/(1-r_1)^2)$. It is quite easy to establish that it takes each such value only once, although this is not of too great importance. It should be observed that the functions $f_{Z^*}(z)$ carry real values into real values and thus have real coefficients in their Taylor expansions about $z=0$.

Let now $f^*(z)$ be a particular function in this subclass of S and let $f(z) \in S$ be such that $|f(-r_1)| = |f^*(-r_1)|$. Then we have

THEOREM 3. *$|f(r_2)| \leqq |f^*(r_2)|$ for $0 < r_2 < 1$.*

It is readily seen to cause no loss of generality to assume $f(z)$ analytic on $|z|=1$. This function then maps $|z|<1$ on a domain, D' say, in the w'-plane bounded by an analytic curve. We can think of this as the image of E_w by a function $w'(w)$ whose Taylor expansion at $w=0$ begins as follows: $w'(w) = w + aw^2 + \cdots$.

This function maps L on a small Jordan curve L' about $w'=0$ again nearly circular and maps R_1 and R_2 onto respectively conformally equivalent quadrangles R_1' and R_2' in the w'-plane. The segment w_2w_3 goes into a curve S_1 running from 0 to $f(r_2)$ and the segment w_1w_2 goes into a curve S_2 running from $f(-r_1)$ to 0.

Suppose now we had $|f(r_2)| > |f^*(r_2)|$. Let M be a small circle, centre the origin $w'=0$, containing L' on the closed disc it bounds but touching L' from outside. If L was small enough originally, M will meet each image by $w'(w)$ of a curve of $\mathfrak{C}_1^*$ or $\mathfrak{C}_2^*$ in just two points, one near each end. Thus M will cut off from R_1' (R_2') two domains, one at each end, leaving a quadrangle R_1'' (R_2'') with a pair of opposite sides on M and the other sides lying along the corresponding sides of R_1' (R_2'). The modules M_1'', M_2'' of R_1'', R_2'' for the classes of curves joining the pair of sides lying on M in each case exceed the quantities M_1, M_2 which were the modules for R_1', R_2' by conformal invariance.

Let u_1 be a function harmonic in R_1'', continuous on its closure, with $\partial u_1/\partial n = 0$ on M, $u_1 = 1$ on S_1 and $u_1 = 0$ on the remainder of the boundary of R_1''. Let u_2 be a function harmonic in R_2'', continuous on its closure, with $\partial u_2/\partial n = 0$ on M, $u_2 = -1$ on S_2 and $u_2 = 0$ on the remainder of the boundary of R_2''. Let $u'' = u_1$ on R_1'' and its boundary, $u'' = u_2$ on R_2'' and its boundary. Let $|w'| \leqq R$ be a circle large enough to contain D' and extend u'' to vanish on the portion of this circle outside of D'.

In this way u'' is defined and continuous on the circular ring. Consider the surface $u''(\xi, \eta)$ ($w' = \xi + i\eta$) over this ring. We apply to it circular

symmetrization with respect to the half-plane through the positive ξ-axis and perpendicular to the (ξ, η) plane. In this way we obtain a function $\tilde{u}(\xi, \eta)$ still defined over the circular ring. The set on which $1 > \tilde{u}(\xi, \eta) > 0$ is a quadrangle $\boldsymbol{R}_1$ with a pair of opposite sides on M and one further side along a segment Σ_1 of the positive real axis from M to a value $\geqq |f(r_2)|$, described twice. $\tilde{u} = 1$ on the latter and $= 0$ on the last side of $\boldsymbol{R}_1$. The set on which $0 > \tilde{u}(\xi, \eta) > -1$ is a quadrangle $\boldsymbol{R}_2$ disjoint from $\boldsymbol{R}_1$ with a pair of opposite sides on M and one further side along a segment Σ_2 of the negative real axis from a value $\leqq -|f(-r_1)|$ to M. $\tilde{u} = -1$ on the latter and $= 0$ on the last side of $\boldsymbol{R}_2$.

Let the modules of $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ corresponding to M_1'', M_2'' be $\boldsymbol{M}_1$, $\boldsymbol{M}_2$. Then $1/\boldsymbol{M}_i \leqq D(\tilde{u}; \boldsymbol{R}_i) \leqq D(u''; R_i'') = 1/M_i''$ for $i = 1, 2$.

The first inequality follows from Lemma 2 and the Dirichlet principle, the second follows by the usual symmetrization argument, the final equality follows by Lemma 2. Thus $\boldsymbol{M}_i \geqq M_i'' \geqq M_i$.

Consider now the domain $\mathfrak{F}$ bounded by M, Σ_1 and Σ_2 with the point at infinity regarded as a distinguished interior point. For it we regard the module problem analogous to that which in **3** defined the module $M^*(a_1, a_2)$. Here the curves of the class corresponding to $\mathfrak{C}_i$ run from M back to M and separate Σ_i from the point at infinity. The existence of the extremal metric in this case follows from the symmetry of the configuration with respect to the real axis as in [2], p. 348. The real axis divides the configuration into two symmetric halves and the corresponding problem for each is a pentagon problem. If P_1 denotes the portion of the real axis to the right of Σ_1 and P_2 denotes the portion of the real axis to the left of Σ_2, then the upper half of M, Σ_1, P_1, P_2, Σ_2 play the roles of 12, 23, 34, 45, 51. Thus Γ_1 is the class of curves joining M and P_1, Γ_2 the class of curves joining M and P_2. The appropriate auxiliary conditions are that, in the metric $\rho\,|\,dw'\,|$, curves of Γ_1 shall have length at least $\frac{1}{2}a_1$, curves of Γ_2 length at least $\frac{1}{2}a_2$. Let the module for $\mathfrak{F}$ given by this problem be denoted by $\boldsymbol{M}^*(a_1, a_2)$.

Just as in the proof of Lemma 3 we have at once that

$$\boldsymbol{M}^*(a_1, a_2) \geqq a_1^2 \boldsymbol{M}_1 + a_2^2 \boldsymbol{M}_2 \geqq a_1^2 M_1 + a_2^2 M_2 = M^*(a_1, a_2).$$

We will now show that this is in contradiction to the assumption $|f(r_2)| > |f^*(r_2)|$ provided that L be chosen small enough to begin with.

We will consider the two corresponding pentagon problems: the pentagon Π_1 having as sides the upper halves of L, s_{12}, s_{22} and the portions of the real axis to the right of s_{12} and to the left of s_{22}, the second pentagon Π_2 having as sides the upper halves of M, Σ_1, Σ_2, P_1, P_2. The function ζ maps

Π_1 on the canonical domain associated with the values $\frac{1}{2}a_1$, $\frac{1}{2}a_2$. In it the Euclidean metric is the extremal metric. Assuming Π_2 to lie also in the w-plane, ζ maps Π_2 on a subdomain of the preceding canonical domain differing from it in the following manner: whereas L goes into a vertical segment (using the normalization of ζ given in **3**), M goes into a curve joining the same horizontal lines and lying arbitrarily close to the segment when L is chosen small enough; also whereas s_{12} goes into a horizontal side, the image of Σ_1 covers this side and overlaps on the adjacent vertical side, and whereas s_{22} goes into a horizontal side, the image of Σ_2 covers this side and may overlap on the adjacent vertical side.

Now let us take in the above canonical domain a fixed vertical segment σ joining the images of s_{12} and s_{22} and lying to the left of the images of L and M. This cuts off to its left from the image of Π_1 a pentagon $\Pi_1^{(0)}$ which is in canonical form corresponding to certain numbers $a_1^{(0)}$, $a_2^{(0)}$. Let the value of the corresponding module be $M^{(0)}$. From Π_2, σ cuts off to its left a pentagon $\Pi_2^{(0)}$ whose module $\boldsymbol{M}^{(0)}$ for these numbers $a_1^{(0)}$, $a_2^{(0)}$ is strictly smaller than $M^{(0)}$. To the right of σ lies in the image of Π_1 a rectangle whose other vertical side is the image of L. Let the vertical dimension of the rectangle be l, its horizontal dimension λ. Then $\frac{1}{2}a_1 = a_1^{(0)} + \lambda$, $\frac{1}{2}a_2 = a_2^{(0)} + \lambda$. If L has been chosen small enough the image of Π_2 will contain to the right of σ a rectangle N of horizontal dimension $\lambda - \epsilon$ for any preassigned $\epsilon > 0$.

Now for the pentagon problems for Π_1 and Π_2 corresponding to the values $\frac{1}{2}a_1$, $\frac{1}{2}a_2$ and taken in their images by ζ we use the following metrics. For the image of Π_1, the Euclidean metric is the extremal metric and gives $\frac{1}{2}M^*(a_1, a_2) = M^{(0)} + l\lambda$.

For the image of Π_2 we take in $\Pi_2^{(0)}$ the extremal metric yielding the module $\boldsymbol{M}^{(0)}$ and to the right of σ we take in N the expanded Euclidean metric $\lambda(\lambda - \epsilon)^{-1} \mid d\zeta \mid$ and elsewhere the metric 0. This metric is at once seen to be admissible for the pentagon problem for Π_2 corresponding to the values $\frac{1}{2}a_1$, $\frac{1}{2}a_2$ and thus gives at least the value $\frac{1}{2}\boldsymbol{M}^*(a_1, a_2)$ but then, for ϵ small enough,

$$\tfrac{1}{2}\boldsymbol{M}^*(a_1, a_2) \leqq \boldsymbol{M}^{(0)} + (\lambda/(\lambda - \epsilon))^2(\lambda - \epsilon)l < M^{(0)} + \lambda l + 2l\epsilon < \tfrac{1}{2}M^*(a_1, a_2).$$

This contradicts the previous deduction that $\frac{1}{2}M^*(a_1, a_2) \leqq \frac{1}{2}\boldsymbol{M}^*(a_1, a_2)$ and shows that our assumption $|f(r_2)| > |f^*(r_2)|$ was false, proving the theorem.

5. THEOREM 4. *If* $f(z) \in S$, *and* $0 < r_1 \leqq r_2 < 1$, *then*

$$|f(-r_1)| + |f(r_2)| \leqq r_2/(1-r_2)^2 + r_1/(1+r_1)^2,$$

equality being attained for the function $z(1-z)^{-2}$.

Indeed, by Theorem 3, the maximum of $|f(-r_1)| + |f(r_2)|$ for r_1, r_2 fixed and f varying in the family S is attained for a function with real coefficients. For such

$$|f(-r_1)| + |f(r_2)| = r_2 + b_2 r_2^2 + \cdots + r_1 - b_2 r_1^2 + \cdots.$$

For functions with real coefficients we have $|b_n| \leqq n$ and since $r_1 \leqq r_2$

$$\begin{aligned} |f(-r_1)| + |f(r_2)| &\leqq r_2 + 2r_2^2 + 3r_2^3 + \cdots + r_1 - 2r_1^2 + 3r_1^3 + \cdots \\ &\leqq r_2/(1-r_2)^2 + r_1/(1+r_1)^2. \end{aligned}$$

The statement concerning equality is evident.

This theorem extends a result of Golusin [1], obtained by an entirely different method, namely, Löwner's parametric method.

COROLLARY 1. *If* $f(z) \in S$, $0 < r_1 \leqq r_2 < 1$, *then*

$$|f(-r_1 e^{i\theta})| + |f(r_2 e^{i\theta})| \leqq r_2/(1-r_2)^2 + r_1/(1+r_1)^2,$$

equality being attained for the function $z(1-ze^{-i\theta})^{-2}$.

COROLLARY 2. *If* $f(z) \in S$, $0 < r < 1$ *then, for* $|z_1| = r$,

$$|f(-z_1)| + |f(z_1)| \leqq 2r(1+r^2)/(1-r^2)^2,$$

equality being attained for the function $z(1-ze^{-i\theta})^{-2}$ *with* $\theta = \arg z_1$.

COROLLARY 3. *If* $f(z) \in S$, *then* $|b_3| \leqq 3$.

Indeed, if $z = re^{i\theta}$, then

$$|f(-z) + f(z)| \leqq |f(-z)| + |f(z)| \leqq 2r(1+r^2)/(1-r^2)^2.$$

Thus $r(1 + \Re(b_3 z^2) + O(r^4)) \leqq r(1 + 3r^2 + O(r^4))$ or $\Re(b_3 e^{2i\theta}) \leqq 3$.

Proper choice of θ gives $|b_3| \leqq 3$.

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THE NUMBER OF IRREDUCIBLE REPRESENTATIONS OF SIMPLE RINGS WITH NO MINIMAL IDEALS.*

By ALEX ROSENBERG.[1]

1. Introduction. Jacobson has called an associative ring primitive if it admits a faithful irreducible module [4]. He also showed that in case the ring possessed minimal one-sided ideals this module was unique up to an isomorphism. By examples, both Jacobson [4] and Kurosh [8] showed that the uniqueness need no longer hold if the hypothesis of a minimal one-sided ideal be dropped.

In this paper we show that a large class of simple rings with no minimal one-sided ideals always admit an infinite number of non-isomorphic irreducible modules. This is true, for example, for any simple algebraic alegbra of countable dimension with no minimal one-sided ideal and unit, and for the simple homomorph of the ring of all linear transformations on an infinite dimensional vector space.

In the last part of the paper we use our methods to complete the solution of a problem recently attacked by M. A. Naĭmark [9]. Using his results we are able to show that the ring of all completely continuous operators on a separable Hilbert space is characterized by the fact that it is a C*-algebra with a unique irreducible representation.

I should like to take this opportunity to thank Professors I. Kaplansky and D. Zelinsky for several conversations with regard to this paper. It was through the former that I became aware of Naĭmark's work.

2. Isomorphic irreducible modules. Let A be an associative ring with unit and $\mathfrak{M}$ a unitary irreducible right A-module. That is, $\mathfrak{M}$ is an additive abelian group admitting the elements of A as right operators; for all x in $\mathfrak{M}$, $x \cdot 1 = x$, 1 the unit of A; and the only submodules of $\mathfrak{M}$ are 0 and $\mathfrak{M}$. The homomorphism of A onto the ring of endomorphisms that A induces on $\mathfrak{M}$ is called an irreducible representation of A; if the homomorphism happens to be one-to-one, we speak of a faithful irreducible representation. The set

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[1] Part of this work was carried out while the author was under contract to the U. S. Air Force at the University of Michigan.

of all endomorphisms of $\mathfrak{M}$ that commute with those induced by A form a division ring D. Hence $\mathfrak{M}$ may be thought of as a vector space over D and the endomorphisms induced by A as linear transformations on $\mathfrak{M}$. It is known that this ring of linear transformations is dense in $\mathfrak{M}$ over D, Theorem 6 in [4].

Two right modules $\mathfrak{M}$ and $\mathfrak{M}'$ are said to be isomorphic if there exists a one-to-one mapping σ of $\mathfrak{M}$ onto $\mathfrak{M}'$ such that $\sigma(xa) = \sigma(x)a$, x in $\mathfrak{M}$, a in A. It is well known that every irreducible right A-module is isomorphic to a quotient module A/J, where J is a maximal right ideal in A.[2] We then have

LEMMA 1. *Let A be a ring with unit. Let J and J' be two maximal right ideals in A. Then the irreducible right modules A/J and A/J' are isomorphic if and only if there is an element a in A but not in J' such that $aJ \subset J'$.*

Proof. If an isomorphism σ exists we set $\sigma(1 + J) = a + J'$. If an element a with the above properties exists, we define a mapping σ by $\sigma(x + J) = ax + J'$. It is easily seen that σ is a module homomorphism onto, with a kernel consisting of all residue classes $x + J$ such that ax lies in J'. However, the set of x with this property is a proper right ideal and so coincides with J. Thus σ is the desired isomorphism.

As an immediate consequence we have

COROLLARY 1. *Let A be a ring with unit. Then the cardinal of the set of maximal right ideals, which give rise to irreducible right modules isomorphic to a given one, is at most the cardinal of the ring.*

Proof. Let the given irreducible module be isomorphic to A/J'. Then for each J such that $A/J \cong A/J'$ there is an element a in A such that $aJ \subset J'$, a not in J'. Moreover different J's give rise to different a's, for otherwise $aA \subset J'$.

The lemma may be reformulated as follows: For convenience sake we stick to the case of a simple ring, that is one where all the irreducible representations are faithful.[3] If A is thought of as a ring of linear transformations on A/J', the maximal right ideal J' consists of all those linear transformations annihilating the vector $x = 1 + J'$. If J is a maximal right ideal such that $A/J \cong A/J'$, we see that J is the annihilator of the vector xa.

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[2] Since A has a unit, Zorn's lemma yields the existence of maximal right ideals.

[3] A ring with unit is simple if the only two-sided ideals are 0 and the ring itself.

Conversely, let J be the annihilator of some vector $y \neq 0$ in A/J'. Since A/J' is irreducible there is an element a in A such that $xa = y$, then $aJ \subset J'$ and $A/J \cong A/J'$. Thus

LEMMA 2. *Let J be a maximal right ideal in a simple ring wtih unit A. Then all the irreducible right modules isomorphic to A/J arise from maximal right ideals which are annihilators of vectors in A/J.*

3. The ring $L - F$. Let $\mathfrak{V}$ be a vector space of dimension $\aleph \geqq \aleph_0$ over a division ring D.[4] We denote by L the ring of all linear transformations on $\mathfrak{V}$. It is known that L has a maximal two-sided ideal F, the ideal of all linear transformations whose ranges have dimension $< \aleph$; cf. [5], Theorem 2. Thus $A = L - F$ is a simple ring with unit, whose irreducible representations we now study.

The ring L may be thought of as the ring of all row finite $\aleph \times \aleph$ matrices with elements in D. Thus, as a set, L is a subset of the set of all functions from a set with $\aleph$ elements to D. Hence if the cardinal of $D = \mathfrak{d}$, $\mathfrak{l} =$ the cardinal of L, $\mathfrak{l} \leq \mathfrak{d}^{\aleph}$. In particular, if we assume that $\mathfrak{d} \leq \exp \aleph$,[5]

$$\mathfrak{l} \leq \mathfrak{d}^{\aleph} \leq (\exp \aleph)^{\aleph} = \exp \aleph^2 = \exp \aleph.$$

We shall now exhibit $\exp \exp \aleph$ distinct maximal right ideals in $A = L - F$. and so by Corollary 1 get the desired number of irreducible representations of A.

Let $B = \{x_\alpha\}$ be any basis of $\mathfrak{V}$. We consider the Boolean algebra $\mathfrak{P}(B)$, of all subsets of B. To each element S of $\mathfrak{P}(B)$ we associate the element e_S of L by setting

$$x_\alpha e_S = x_\alpha, \quad x_\alpha \text{ in } S, \quad x_\alpha e_S = 0, \quad x_\alpha \text{ not in } S.$$

It is easily seen that

$$e_S + e_T - e_S e_T = e_{S \cup T}, \qquad e_S e_T = e_T e_S = e_{T \cap S},$$

so that the e_S form a Boolean algebra isomorphic to $\mathfrak{P}(B)$. They can also be seen to be a maximal commutative set of idempotents of L. Now by a theorem of Pospisil [10] $\mathfrak{P}(B)$ contains $\exp \exp \aleph$ maximal meet ideals containing the ideal of all subsets with cardinal $< \aleph$, which we denote by $\mathfrak{F}$. Let $\mathfrak{M}$ be such a maximal ideal in $\mathfrak{P}(B)$. We let M be the right ideal generated in L by the e_S with S in $\mathfrak{M}$. It is easily seen that M is proper, but even more is true:

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[4] That is, a Hamel basis of V has $\aleph$ elements.

[5] We write $\exp \aleph$ for the $\aleph$-th power of 2.

LEMMA 3. *The right ideal $M + F$ is proper in L.*

Proof. Suppose

$$1 = \sum_1^n e_{S_i} a_i + a, \quad S_i \text{ in } M, \quad a \text{ in } F. \tag{1}$$

Then if T is the complement of $\bigcup_1^n S_i$ in B, T does not lie in $\mathfrak{M}$. By multiplying (1) on the left by e_T, we get $e_T = e_T a$. Thus e_T is in F, T is in $\mathfrak{J} \subset \mathfrak{M}$, contradiction.

By Zorn's lemma we may then embed $M + F$ in a maximal right ideal J. Furthermore the maximal right ideals obtained in this manner from distinct ideals $\mathfrak{M}$ are also distinct. For suppose that one maximal right ideal J contained M_1 and M_2. Then for S in $\mathfrak{M}_1$, T in $\mathfrak{M}_2$ we have $B = S \cup T$, so that 1 is in $M_1 + M_2 \subset J$.

We have thus established the existence of $\exp\exp \aleph$ maximal right ideals in $A = L - F$. Since A has at most $\exp \aleph$ elements, it has at most $\exp\exp \aleph$ right ideals and so using Corollary 1 we get

THEOREM 1. *Let $\mathfrak{V}$ be a vector space of dimension $\aleph \geq \aleph_0$ over a division ring D. Let L be the ring of all linear transformations on $\mathfrak{V}$ and let F be the maximal two sided ideal of L, consisting of all linear transformations whose ranges have dimensions $< \aleph$. If the cardinal of $D \leq \exp \aleph$, the simple ring $L - F$ has exactly* $\exp\exp \aleph$ *non-isomorphic irreducible representations.*

A very similar theorem holds if L is taken to be the ring of all bounded operators on a Hilbert space. For simplicity's sake we stick to the case of a separable Hilbert space. In that case it is known that the set of all completely continuous operators forms the maximal two-sided ideal F in L, and indeed F is the only closed ideal [3], Theorem 1. 4. Thus the ring $L - F$ is simple and contains a unit. Then just as before it can easily be seen that L has at most $\exp \aleph_0$ elements; and a maximal normal orthogonal set of vectors of the Hilbert space again yields $\exp\exp \aleph_0$ maximal right ideals of L containing F. Thus

THEOREM 2. *Let L be the ring of all bounded operators on a separable Hilbert space, F the set of all completely continuous operators. Then the simple ring $L - F$, considered purely algebraically, has exactly* $\exp\exp \aleph_0$ *non-isomorphic irreducible representations.*

Since a simple ring with a minimal one-sided ideal has a unique irreducible representation, it follows that in both cases $L - F$ is a simple ring with no minimal one-sided ideal.

4. Simple algebras of dimension $\aleph_0$. We now consider a simple algebra A with no minimal one-sided ideal over a field Φ. In order to ensure an adequate supply of idempotents we suppose that A is Zorn, i. e., *that every non-nil right ideal contains a nonzero idempotent.*[6] We shall further assume that A has a unit and that the dimension of A over $\Phi = \aleph_0$. We now proceed to construct enough maximal right ideals in A to ensure that A has $\exp \aleph_0 = \mathfrak{c}$ non-isomorphic irreducible representations. By Theorem 2.1 of [7], A contains $\aleph_0$ orthogonal idempotents. Using Zorn's lemma, these may be embedded in a maximal commutative set of idempotents $\mathfrak{B}$.

LEMMA 4. *$\mathfrak{B}$ has $\aleph_0$ elements.*

Proof. Using the right regular representation we can think of A as an algebra of linear transformations on a vector space $\mathfrak{M}$ of dimension $\aleph_0$ over Φ. Now if e and f are in $\mathfrak{B}$, $\mathfrak{M}ef = \mathfrak{M}fe$ so that $\mathfrak{M}ef \subset \mathfrak{M}e$. By using the Peirce decomposition we see that $\mathfrak{M}ef$ is a proper subspace unless $ef = 0$ or $e(1-f) = 0$. Since there are at most $\aleph_0$ orthogonal idempotents in A, and any strictly descending chain of subspaces of $\mathfrak{M}e$ has $\aleph_0$ elements, the lemma is proved.

By defining $e \cup f = e + f - ef$, $e \cap f = ef$ we turn $\mathfrak{B}$ into a Boolean algebra with $\aleph_0$ elements and unit. $\mathfrak{B}$ then has the natural ordering $e \leq f$ if and only if $ef = e$. Then

LEMMA 5. *$\mathfrak{B}$ has no minimal elements.*

Proof. If e is a minimal element of $\mathfrak{B}$ we consider the ring eAe. If h were an idempotent of $eAe \neq e$, it would follow that h would be in $\mathfrak{B}$ too. Thus eAe has only one idempotent. But eAe is Zorn, too; [11], Lemma 2; hence eAe is a division ring and eA a minimal right ideal, contradiction.

By Stone's representation theorem it follows that $\mathfrak{B}$ is isomorphic to the Boolean algebra of all open and closed subsets of a totally disconnected compact Hausdorff space $\mathfrak{S}$ with no isolated points; [13], Theorem 1.6. Since $\mathfrak{B}$ has $\aleph_0$ elements it follows that $\mathfrak{S}$ satisfies the second axiom of countability and so by the Uryson metrization theorem, it is a metric space. But this implies that $\mathfrak{S}$ is homeomorphic to the Cantor set [1], Satz VI′, p. 121. Thus $\mathfrak{S}$ has $\mathfrak{c}$ points and $\mathfrak{B}$ has $\mathfrak{c}$ distinct maximal ideals. Moreover the Chinese Remainder Theorem applies to them, so that if $\mathfrak{M}_1, \cdots, \mathfrak{M}_{n+1}$ are maximal ideals of $\mathfrak{B}$ there is an element of $\mathfrak{B}$ in $\bigcap_1^n \mathfrak{M}_i$ but not in $\mathfrak{M}_{n+1}$.

[6] For a discussion of Zorn rings cf. [7, p. 63].

Just as in 2, we now obtain $\mathfrak{c}$ distinct maximal right ideals J of A which also have the Chinese Remainder Theorem property: $\bigcap_1^n J_i + J_{n+1} = A$. Thus if, in any irreducible representation of A, the J's are annihilators of vectors, these vectors must be linearly independent: for if x_i is annihilated by J_i and $\Sigma\alpha_i x_i = 0$, there is an element of A annihilating $x_1, \cdots, x_n$, but not x_{n+1}. However, since the dimension of A is $\aleph_0$ the dimension of any irreducible module $\mathfrak{M}$ over Φ is also $\aleph_0$, and since the commuting ring of endomorphisms D contains Φ we have that dimension of $\mathfrak{M}$ over $D = \aleph_0$. Thus it follows from Lemma 2 that at most $\aleph_0$ of the modules A/J can be isomorphic to a given one and we have

THEOREM 3. *Let A be a simple Zorn algebra of dimension $\aleph_0$. Suppose that A has a unit and no minimal one-sided ideal. Then A has at least $\mathfrak{c}$ non-isomorphic irreducible representations.*

COROLLARY 2. *Let A be a simple Zorn ring with unit, and suppose A is of dimension $\aleph_0$ over its center. Then A either satisfies the descending chain condition on right ideals or it has $\mathfrak{c}$ non-isomorphic irreducible representations.*

For if A has a minimal one-sided ideal, the presence of a unit implies the descending chain condition; [4], p. 234.

Remarks. The commuting division rings of the irreducible modules are homomorphs of subrings of A; [4], p. 236. Thus is A is an algebraic algebra over an algebraically closed field Φ, e. g., an infinite Kronecker product of 2×2 matrices with complex entries [8], these division rings are all isomorphic to Φ. Thus the representations may be non-isomorphic even if the division rings are isomorphic.

We should also like to point out that the restriction to dimension $\aleph_0$ is essential in our proof. In fact we have not succeeded in proving the existence of $\exp \aleph$ maximal right ideals in A for arbitrary $\aleph$.

5. C^*-algebras with unique irreducible representations. In a recent paper [9] Naĭmark raises the following question: Let A be an irreducible C^*-algebra of bounded operators on a separable Hilbert space $\mathfrak{H}$. Suppose A admits a unique irreducible representation,[7] does this imply that A is the

[7] A representation here is a *-preserving homomorphism into the algebra of all bounded operators on a Hilbert space. Irreducibility means no invariant closed subspaces and uniqueness means that all representations are unitarily equivalent.

algebra of all completely continuous operators on $\mathfrak{H}$? Naĭmark was not able to settle the question completely; however, he did show

LEMMA 6.[8] *Every maximal commutative subalgebra of A has at most $\aleph_0$ regular maximal ideals.*

Since every closed self-adjoint commutative subalgebra of A may be embedded in a maximal one, we see by the structure theory of commutative B^*-algebras that every closed self-adjoint commutative subalgebra has at most $\aleph_0$ regular maximal ideals. Hence the space of regular maximal ideals of such algebras is a locally compact Hausdorff space with at most $\aleph_0$ points. Now

LEMMA 7. *A locally compact Hausdorff space with at most $\aleph_0$ points, contains isolated points.*[9]

Proof. By Théorème 1, § 5 of [2] such a space is of the second category. But a limit point is a set of the first category; thus isolated points exist.

But the existence of isolated points in the space of maximal ideals implies the existence of projections (self-adjoint idempotents). Hence there exists a non-empty maximal commuting set of projections B in A. Now B generates a self-adjoint closed subalgebra of A. The space of regular maximal ideals of this algebra contains at least one isolated point P. If we now take the function e which is 1 at P and 0 everywhere else, it follows that e is a minimal element of B, i. e., for every f in B, $ef = 0$ or e.

LEMMA 8. *The right ideal eA is a closed minimal ideal.*

Proof. The closure is clear. Let x be an element of eA and consider the closed subalgebra generated by xx^*. This subalgebra is in eA and by Lemmas 6 and 7 contains a projection $f = ef \neq 0$. Then $0 \neq fe$ lies in B, and since e is minimal $fe = e$. Hence $xA \supset feA = eA$, so that eA is minimal.

But then by Theorem 7.3 of [6] it follows that A contains the ideal F of all completely continuous operators on $\mathfrak{H}$. Now if $A \neq F$, $A - F$ would still be a C^*-algebra ([6], Theorem 7.2) and so would admit an irreducible representation [12]. This would yield an irreducible representation of A annihilating the ideal F which contradicts the uniqueness of the irreducible representations of A. Hence

[8] Naĭmark assumes that A has a unit and exactly two non-equivalent irreducible representations. However, his proofs can easily be modified to apply to the case where no unit is assumed.

[9] This lemma was pointed out to me by Professor Kaplansky.

THEOREM 4. *Let A be an irreducible C^*-algebra of bounded operators on a separable Hilbert space $\mathfrak{H}$. Suppose that A has a unique irreducible representation. Then A is the algebra of all completely continuous operators on $\mathfrak{H}$.*

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FRACTIONAL INTEGRATION.*

By I. I. HIRSCHMAN, JR.[1]

1. 1. Let $u(\theta) \in L^p(0, 2\pi)$ and have mean value zero, so that

$$u(\theta) \sim \Sigma' a_n e^{in\theta},$$

where $-\infty < n < \infty$ but $n \neq 0$. The fractional integral $u_\alpha(\theta)$, of order α, of $u(\theta)$, is defined by $u_\alpha(\theta) \sim \Sigma' a_n (in)^{-\alpha} e^{in\theta}$, where

$$(in)^{-\alpha} = |\, n \,|^{-\alpha} \exp(\alpha\pi i \operatorname{sgn} n/2).$$

Many interesting properties have been demonstrated in connection with fractional integration particularly by Hardy and Littlewood, see [1]-[4]. The present paper is devoted to certain results for fractional integration which are related to the work of Littlewood and Paley [5], Zygmund [8], [9] and Marcinkiewicz [6], [7]. We prove here that if $1 < p < \infty$, and $0 < \alpha < 1$, then

$$(1) \qquad A' \int_0^{2\pi} |\, u(\theta) \,|^p \, d\theta \leqq \int_0^{2\pi} d\theta \Big[\int_0^{2\pi} |\, u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau) \,|^2 \tau^{-2\alpha-1} \, d\tau \Big]^{\frac{1}{2}p}$$

$$\leqq A'' \int_0^{2\pi} |\, u(\theta) \,|^p \, d\theta,$$

where A' and A'' are positive constants depending only on p and α. Let

$$u(\rho, \theta) \sim \sum_{-\infty}^{\infty}{}' a_n \rho^{|n|} e^{in\theta}, \qquad u_\alpha(\rho, \theta) \sim \sum_{-\infty}^{\infty}{}' a_n (in)^{-\alpha} \rho^{|n|} e^{in\theta}.$$

We also show that if $1 < p < \infty$, $-\infty < \alpha < 1$, then

$$(2) \qquad A' \int_0^{2\pi} |\, u(\theta) \,|^p \, d\theta \leqq \int_0^{2\pi} d\theta \Big[\int_0^{2\pi} (1 - \rho)^{1-2\alpha} \,|\, u_{\alpha-1}(\rho, \theta) \,|^2 \, d\rho \Big]^{\frac{1}{2}p}$$

$$\leqq A'' \int_0^{2\pi} |\, u(\theta) \,|^p \, d\theta.$$

The relations (1) and (2) are analogous $[u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau)]/\tau$ in (1) corresponding to $u_{\alpha-1}(\rho, \theta)$ in (2). Hardy and Littlewood have shown in [1], that $u \in L^p(0, 2\pi)$ implies $\| u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau) \|_p = o(|\, \tau \,|)^\alpha$, $\tau \to 0$.

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Using the above results we get some information as to the size of "o"; if $1 < p < \infty$, then

$$\int_0^{2\pi} \| u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) \|_p^r \, \tau^{-1-\alpha r} d\tau \leqq A' \| u \|_p^r, \quad r \geqq \operatorname{Max}(2, p);$$

$$\int_0^{2\pi} \| u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) \|_p^r \, \tau^{-1-\alpha r} d\tau \geqq A'' \| u \|_p^r, \quad r \leqq \operatorname{Min}(2, p).$$

1.2. We collect here certain results which we shall need.

THEOREM 1.2a. *Let $u(\theta) \in L^p(0, 2\pi)$, $1 < p < \infty$. We suppose $u(\theta)$ extended by periodicity to the entire real axis. Let*

$$u^\dagger(\theta) = \text{l. u. b.}_h \, [1/h \int_\theta^{\theta+h} | u(t) | \, dt].$$

Then $\| u^\dagger(\theta) \|_p \leqq A \| u \|_p$, where A is a constant depending only on p.

This result is due to Hardy and Littlewood. A proof may be found in Zygmund [10], pp. 241-250.

Let $P(\rho, \theta)$ be the Poisson kernel

$$P(\rho, \theta) = \tfrac{1}{2\pi}(1-\rho^2)/(1+2\rho\cos\theta+\rho^2).$$

THEOREM 1.2b. *Let $u(z) = \sum_1^\infty a_n z^n \in H^p(0, 2\pi)$ and*

$$\chi(\rho, \theta) = \{ \int_0^{2\pi} | u'(\rho e^{i\tau}) |^2 P(\rho, \theta-\tau) d\tau \}^{\frac{1}{2}}, \quad g^\dagger(\theta) = \{ \int_0^1 (1-\rho)\chi^2(\rho, \theta) d\rho \}^{\frac{1}{2}}.$$

Then $\| g^\dagger(\theta) \|_p \leqq A \| u \|_p$ for $1 < p < \infty$, where A is a constant depending only on p.

THEOREM 1.2c. *Let $u(z) = \sum_1^\infty a_n z^n \in H^p(0, 2\pi)$, $\alpha > 0$ and*

$$S(\theta) = \{ \int_0^1 \int_{|\tau-\theta| \leqq \alpha(1-\rho)} | u'(\rho e^{i\tau}) |^2 \rho \, d\tau \, d\rho \}^{\frac{1}{2}}.$$

Then $\| S(\theta) \|_p \leqq A \| u \|_p$ for $1 < p < \infty$, where A is a constant depending only on α and p.

THEOREM 1.2d. *Let $u(z) = \sum_1^\infty a_n z^n \in H^p(0, 2\pi)$, $\alpha > 0$, $\sigma > 1$ and*

$$G_\sigma(\theta) = \int_0^1 \int_{|\tau-\theta| \geqq \alpha(1-\rho)} | u'(\rho e^{i\tau}) |^2 (1-\rho)^\sigma \, | \tau - \theta |^{-\sigma} d\tau d\rho.$$

Then $\| G_\sigma(\theta) \|_p \leqq A \| u \|_p$ *for* $1 < p < \infty$, *where* A *is a constant depending only on* α, σ, *and* p.

These results are due to Littlewood and Paley, and Zygmund; see [5] and [8].

It should be pointed out that the deeper difficulties of the present subject are already overcome when these results are assumed. Many of the demonstrations here consist in making complicated though elementary reductions until a point is reached where it is possible to apply these theorems.

2.1. If $u \in L^p(0, 2\pi)$, $1 < p < \infty$, and if u has mean value zero, we define

$$\Delta(\alpha, u, \theta) = [\int_0^1 (1-\rho)^{1-2\alpha} \mid u_{\alpha-1}(\rho, \theta) \mid^2 d\rho]^{\frac{1}{2}}.$$

We shall use A for any positive constant, not necessarily the same at each appearance, which depends only on α and p.

THEOREM 2.1. *If* $1 < p < \infty$, $-\infty < \alpha < 1$, *then*

$$\| \Delta(\alpha, u, \theta) \|_p \leqq A \| u(\theta) \|_p. \tag{1}$$

It is sufficient to prove this under the assumption that $u(\rho, \theta)$ is harmonic in a region containing the unit circle in its interior. If this case has been established then, if $0 < \lambda < 1$,

$$\int_0^{2\pi} d\theta [\int_0^1 (1-\rho)^{1-2\alpha} \mid u_{\alpha-1}(\lambda\rho, \theta) \mid^2 d\rho]^{\frac{1}{2}p} \leqq A \int_0^{2\pi} \mid u(\lambda, \theta) \mid^p d\theta.$$

Letting $\lambda \to 1-$ and using Fatou's lemma, (1) follows.

Integrating by parts we find that

$$\Delta(\alpha, u, \theta)^2 = 2(2-2\alpha)^{-1} \int_0^1 (1-\rho)^{2-2\alpha} \mid u_{\alpha-1}(\rho, \theta) \mid d/d\rho \mid u_{\alpha-1}(\rho, \theta) \mid d\rho.$$

Since $(d/d\rho) \mid u_{\alpha-1}(\rho, \theta) \mid$ is equal in absolute value to $\mid (d/d\rho) u_{\alpha-1}(\rho, \theta) \mid$ and $d/d\rho\, u_{\alpha-1}(\rho, \theta) = u^\sim_{\alpha-2}(\rho, \theta)/\rho$, where $u^\sim$ is the conjugate of u, $u^\sim(\theta) = \sum_{-\infty}^{\infty}{}' a_n(-i \operatorname{sgn} n) e^{in\theta}$, we have, by Schwarz's inequality,

$$\Delta(\alpha, u, \theta)^2 \leqq A [\int_0^1 \rho^{-2}(1-\rho)^{1-2\alpha} \mid u_{\alpha-1}(\rho, \theta) \mid^2 d\rho]^{\frac{1}{2}} \Delta(\alpha - 1, u^\sim, \theta).$$

Also $\int_0^1 \rho^{-2}(1-\rho)^{1-2\alpha} \mid u_{\alpha-1}(\rho, \theta) \mid^2 d\rho = I_1 + I_2$, where I_1 is the integral

from 0 to $\frac{1}{2}$ and I_2 from $\frac{1}{2}$ to 1. Using the inequality $|a_n| \leqq (2\pi)^{-1/p} \| u \|_p$, it is easy to verify that $I_1 \leqq A \| u \|_p^2$. Also $I_2 \leqq 4\Delta(\alpha, u, \theta)^2$. Thus

$$\Delta(\alpha, u, \theta)^2 \leqq A[\| u \|_p + \Delta(\alpha, u, \theta)]\Delta(\alpha - 1, u^\sim, \theta). \tag{2}$$

It is easily deduced from this that

$$\Delta(\alpha, u, \theta) \leqq A[\| u \|_p + \Delta(\alpha - 1, u^\sim, \theta)]. \tag{3}$$

In view of this formula, if our theorem has been established for $\alpha - 1$, then it will follow for α. Indeed, suppose that it has been established for $\alpha - 1$ so that $\| \Delta(\alpha - 1, u, \theta) \|_p \leqq A \| u \|_p$. By the theorem of M. Riesz on conjugate functions $\| u^\sim \|_p \leqq A \| u \|_p$. Using these inequalities and (3), it follows that $\| \Delta(\alpha, u, \theta) \|_p \leqq A \| u \|_p$. It is thus sufficient to prove our theorem under the assumption $\alpha < -\frac{1}{2}$.

We have

$$u_{\alpha-1}(\rho, \theta) = \int_0^{2\pi} u_{-1}(\rho^{\frac{1}{2}}, t) k(\rho^{\frac{1}{2}}, \theta - t)\, dt, \qquad k(\rho, \theta) = \tfrac{1}{2\pi} \sum_{-\infty}^{\infty}{}' (in)^{-\alpha} \rho^{|n|} e^{in\theta}.$$

It is easily verified that $|k(\rho, \theta)| \leqq A\, |1 - \rho e^{i\theta}|^{-1+\alpha}$ for $|\rho| < 1$. Thus

$$u_{\alpha-1}(\rho, \theta) \leqq A \int_0^{2\pi} |u_{-1}(\rho^{\frac{1}{2}}, t)|\, |1 - \rho^{\frac{1}{2}} e^{i(\theta - t)}|^{-1+\alpha}\, dt.$$

Using Schwarz's inequality we find that

$$|u_{\alpha-1}(\rho, \theta)|^2 \leqq A \int_0^{2\pi} |u_{-1}(\rho^{\frac{1}{2}}, t)|^2\, |1 - \rho^{\frac{1}{2}} e^{i(\theta - t)}|^{-2}\, dt \int_0^{2\pi} |1 - \rho^{\frac{1}{2}} e^{i(\theta - t)}|^{2\alpha}\, dt.$$

Provided $\alpha < -\frac{1}{2}$, we have $\int_0^{2\pi} |1 - \rho^{\frac{1}{2}} e^{i(\theta - t)}|^{2\alpha}\, dt \leqq A(1 - \rho^{\frac{1}{2}})^{2\alpha+1}$. Since, $P(\rho, t)$ being the Poisson kernel, $(1 - \rho^{\frac{1}{2}})\, |1 - \rho^{\frac{1}{2}} e^{i(\theta - t)}|^{-2} \leqq AP(\rho^{\frac{1}{2}}, \theta - t)$, it follows that

$$|u_{\alpha-1}(\rho, \theta)|^2 \leqq A(1 - \rho^{\frac{1}{2}})^{2\alpha} \int_0^{2\pi} |u_{-1}(\rho^{\frac{1}{2}}, t)|^2 P(\rho^{\frac{1}{2}}, \theta - t)\, dt;$$

$$\Delta(\alpha, u, \theta)^2 \leqq A \int_0^1 (1 - \rho)^{1-2\alpha} (1 - \rho^{\frac{1}{2}})^{2\alpha}\, d\rho \int_0^{2\pi} |u_{-1}(\rho^{\frac{1}{2}}, t)|^2\, P(\rho^{\frac{1}{2}}, \theta - t)\, dt$$

$$\leqq \int_0^1 (1 - r) \int_0^{2\pi} |u_{-1}(r, t)|^2\, P(r, \theta - t)\, dt\, dr.$$

That $\| \Delta(\alpha, u, \theta) \|_p \leqq A \| u \|_p$ is now a consequence of Theorem 1.2b, and M. Riesz's theorem on conjugate functions.

2.2. In this section we will establish the converse inequality.

THEOREM 2.2. *If* $1 < p < \infty$, $-\infty < \alpha < -1$, *then*

$$\| \Delta(\alpha, u, \theta) \|_p \geqq A \| u(\theta) \|_p. \tag{1}$$

Again, it is sufficient to prove this under the assumption that $u(\rho, \theta)$ is harmonic in a region containing the unit circle in its interior. For if it has been proven in this case and if $0 < \lambda < 1$, then

$$\| u(\lambda, \theta) \|_p \leqq A \| \Delta(\alpha, u(\lambda\rho, \theta), \theta) \|_p.$$

Now

$$\Delta(\alpha, u(\lambda\rho, \theta), \theta)^2 = \int_0^1 (1-\rho)^{1-2\alpha} \mid u_{\alpha-1}(\lambda\rho, \theta)|^2 \, d\rho$$

$$= \lambda^{-1} \int_0^\lambda (1-\rho\lambda^{-1})^{1-2\alpha} \mid u_{\alpha-1}(\rho, \theta)|^2 \, d\rho \leqq \lambda^{-1} \Delta(\alpha, u, \theta)^2.$$

Thus we have $\| u(\lambda, \theta) \|_p \leqq A\lambda^{-\frac{1}{2}} \| \Delta(\alpha, u, \theta) \|_p$. Letting $\lambda \to 1$ — we obtain (1).

Let $v(\rho, \theta)$ be harmonic in a region containing the unit circle in its interior and let $v(0, \theta) = 0$. We set $v(\theta) = v(1, \theta)$. Inserting the Fourier expansions of $u(\rho, \theta) v(\rho, \theta)$, etc., it is very easy to verify that

$$\int_0^{2\pi} u(\theta) v(\theta) d\theta$$

$$= \int_0^{2\pi} \int_0^1 (1-\rho)\rho^{-2} [4u_{-1}(\rho, \theta) v_{-1}(\rho, \theta) + 2u_{-1}(\rho, \theta) v^{\sim}(\rho, \theta)] d\rho d\theta.$$

Thus $\int_0^{2\pi} u(\theta) v(\theta) d\theta = \int_0^{2\pi} u_\alpha(\theta) v_{-\alpha}(\theta) d\theta$

$$= \int_0^{2\pi} \int_0^1 (1-\rho)\rho^{-2} [4u_{\alpha-1}(\rho, \theta) v_{-\alpha-1}(\rho, \theta) + 2u_{\alpha-1}(\rho, \theta) v^{\sim}_{-\alpha}(\rho, \theta)] d\rho d\theta,$$

$= I_1 + I_2$, where I_1 corresponds to the range $0 \leqq \theta \leqq 2\pi$, $0 \leqq \rho \leqq \epsilon$ and I_2 to the range $0 \leqq \theta \leqq 2\pi$, $\epsilon \leqq \rho \leqq 1$. Let us now suppose $\| v \|_q \leqq 1$. Given $a > 0$ it is possible to choose $\epsilon > 0$, depending only upon p, α and a, so that $| I_1 | \leqq a \| u \|_p$. There will then exist a constant $A(a)$ such that

$$| I_2 | \leqq A(a) \int_0^{2\pi} \int_0^1 (1-\rho) [4u_{\alpha-1}(\rho, \theta) v_{-\alpha-1}(\rho, \theta) + 2u_{\alpha-1}(\rho, \theta) v^{\sim}_{-\alpha}(\rho, \theta)] d\rho d\theta.$$

By Schwarz's inequality,

$$\left| \int_0^1 (1-\rho) u_{\alpha-1}(\rho, \theta) v_{-\alpha-1}(\rho, \theta) d\rho \right| \leqq \Delta(\alpha, u, \theta) \Delta(-\alpha, v, \theta).$$

By Theorem 2.1 and Hölder's inequality,

$$\left| \int_0^{2\pi} \int_0^1 (1-\rho) u_{\alpha-1}(\rho,\theta) v_{-\alpha-1}(\rho,\theta) d\rho d\theta \right| \leqq A \parallel \Delta(\alpha, u, \theta) \parallel_p.$$

Let $w = v^{\sim}_1$; evidently $\parallel w \parallel_q \leqq A$. Again

$$\left| \int_0^1 (1-\rho) u_{\alpha-1}(\rho,\theta) v^{\sim}_{-\alpha}(\rho,\theta) d\rho \right| \leqq \Delta(\alpha, u, \theta)\Delta(-\alpha, w, \theta),$$

$$\left| \int_0^{2\pi} \int_0^1 (1-\rho) u_{\alpha-1}(\rho,\theta) v^{\sim}_{-\alpha}(\rho,\theta) d\rho d\theta \right| \leqq A \parallel \Delta(\alpha, u, \theta \parallel_p.$$

It follows that $\left| \int_0^{2\pi} u(\theta) v(\theta) d\theta \right| \leqq a \parallel u \parallel_p + A(a) \parallel \Delta(\alpha, u, \theta) \parallel_p$. We have l. u. b. $\left| \int_0^{2\pi} u(\theta) v(\theta) d\theta \right| \geqq \frac{1}{2} \parallel u \parallel_p$, when v varies over the class of functions described above and thus $\frac{1}{2} \parallel u \parallel_p \leqq a \parallel u \parallel_p + A(a) \parallel \Delta(\alpha, u, \theta) \parallel_p$. Since a is arbitrary, this implies our theorem for $\alpha > -1$, the restriction arising from the fact that in our proof we applied Theorem 2.1 with $-\alpha$. The extension to other values of α is easily effected, using (2) of § 2.1.

For $\alpha = 0$ Theorems 2.1 and 2.2 have been established by Littlewood and Paley [5].

3.1. If $u \in L^p(0, 2\pi)$ and if u has mean value zero, we define

$$\delta(\alpha, u, \theta) = \left[\int_0^{2\pi} | u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) |^2 \tau^{-2\alpha-1} d\tau \right]^{\frac{1}{2}}.$$

THEOREM 3.1. *If* $1 < p < \infty$, $0 < \alpha < 1$, *then*

$$\parallel \delta(\alpha, u, \theta) \parallel_p \leqq A \parallel u \parallel_p. \tag{1}$$

Let $v(\theta) = \sum_1^\infty a_n e^{in\theta}$, $w(\theta) = \sum_{-\infty}^{-1} a_n e^{in\theta}$. We have by Riesz's theorem, see [10], pp. 147-151, $\parallel v \parallel_p \leqq A \parallel u \parallel_p$, $\parallel w \parallel_p \leqq A \parallel u \parallel_p$. Further,

$$| u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) |^2 \leqq 2 | v_\alpha(\theta+\tau) - v_\alpha(\theta-\tau) |^2 + 2 | w_\alpha(\theta+\tau) - w_\alpha(\theta-\tau) |^2,$$

and from this it is easily deduced that $\delta(\alpha, u, \theta) \leqq A\delta(\alpha, v, \theta) + A\delta(\alpha, w, \theta)$. Thus if our theorem were demonstrated for v and w, so that

$$\parallel \delta(\alpha, v, \theta) \parallel_p \leqq A \parallel v \parallel_p, \qquad \parallel \delta(\alpha, w, \theta) \parallel_p \leqq A \parallel w \parallel_p,$$

then (1) would follow. Thus we may assume that $u(\rho, \theta) = \sum_1^\infty a_n \rho^n e^{in\theta}$.

We write $u(z)$, $z = \rho e^{i\theta}$ for $u(\rho, \theta)$. Further it is no restriction to suppose that $u(z)$ is analytic in a region containing the unit circle in its interior.

Let $\rho_\tau = 1 - \tau/(4\pi)$. We have

$$u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) = \tfrac{1}{2}\int_{\rho_\tau e^{i(\theta+\tau)}}^{e^{i(\theta+\tau)}} [z - e^{i(\theta+\tau)}]^2 u'''_\alpha(z)\,dz$$

$$+ \tfrac{1}{2}\int_{\rho_\tau e^{i\theta}}^{\rho_\tau e^{i(\theta+\tau)}} [z - e^{i(\theta+\tau)}]^2 u'''_\alpha(z)\,dz$$

$$+ \tfrac{1}{2} u''_\alpha(\rho_\tau e^{i\theta})\{-[\rho_\tau e^{i\theta} - e^{i(\theta+\tau)}]^2 + [\rho_\tau e^{i\theta} - e^{i(\theta-\tau)}]^2\}$$

$$+ u'_\alpha(\rho_\tau e^{i\theta})\{[\rho_\tau e^{i\theta} - e^{i(\theta+\tau)}] - [\rho_\tau e^{i\theta} - e^{i(\theta-\tau)}]\}$$

$$- \tfrac{1}{2}\int_{\rho_\tau e^{i(\theta-\tau)}}^{e^{i(\theta-\tau)}} [z - e^{i(\theta-\tau)}]^2 u'''_\alpha(z)\,dz$$

$$- \tfrac{1}{2}\int_{\rho_\tau e^{i(\theta-\tau)}}^{\rho_\tau e^{i\theta}} [z - e^{i(\theta-\tau)}]^2 u'''_\alpha(z)\,dz,$$

$$u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

Consider I_1. We have

$$|I_1| \leqq \int_{\rho_\tau}^1 |u'''_\alpha(\rho e^{i(\theta+\tau)})|\,(1-\rho)^2 d\rho,$$

$$|I_1|^2 \leqq \int_{\rho_\tau}^1 |u'''_\alpha(\rho e^{i(\theta+\tau)})|^2 (1-\rho)^{5-\alpha} d\rho \int_{\rho_\tau}^1 (1-\rho)^{-1+\alpha} d\rho,$$

$$\leqq A\tau^\alpha \int_{\rho_\tau}^1 |u'''_\alpha(\rho e^{i(\theta+\tau)})|^2 (1-\rho)^{5-\alpha} d\rho.$$

It follows that

$$\int_0^{2\pi} |I_1|^2 \tau^{-2\alpha-1} d\tau \leqq A \int_0^{2\pi} \tau^{-1-\alpha} d\tau \int_{\rho_\tau}^1 |u'''_\alpha(\rho e^{i(\theta+\tau)})|^2 (1-\rho)^{5-\alpha} d\rho.$$

If we define $h_\rho(\tau)$ as 1 for $4\pi(1-\rho) \leqq \tau \leqq 2\pi$ and as 0 for $0 \leqq \tau < 4\pi(1-\rho)$, then

$$\int_0^{2\pi} |I_1|^2 \tau^{-2\alpha-1} d\tau \leqq A \int_{\frac{1}{2}}^1 (1-\rho)^{5-\alpha} d\rho \int_0^{2\pi} |u'''_\alpha(\rho e^{i(\theta+\tau)})|^2 h_\rho(\tau) \tau^{-\alpha-1} d\tau.$$

We have

$$u'''_\alpha(\rho e^{i(\theta+\tau)})$$

$$= \tfrac{1}{2\pi}\int_0^{2\pi} u'(\rho^{\frac{1}{2}} e^{is})\Big[\sum_1^\infty \frac{(n-1)(n-2)}{(in)^\alpha} \rho^{\frac{1}{2}n - 5/2} e^{-2is} e^{i(n-3)(\theta+\tau-s)}\Big] ds.$$

If $\rho \geqq \frac{1}{2}$, then $\left| \sum_1^\infty (n-1)(n-2)(in)^{-\alpha}\rho^{\frac{1}{2}n-5/2}e^{i(n-3)t} \right| \leqq A \mid 1-\rho e^{it} \mid^{-3+\alpha}$, and hence

$$| u'''_\alpha(\rho e^{i(\theta+\tau)}) | \leqq A \int_0^{2\pi} u'(\rho^{\frac{1}{2}}e^{is}) | \mid 1-\rho e^{i(\tau+\theta-s)} \mid^{-3+\alpha} ds,$$

(2)

$$| u'''_\alpha(\rho e^{i(\theta+\tau)}) |^2 \leqq A(1-\rho)^{-3+2\alpha} \int_0^{2\pi} | u'(\rho^{\frac{1}{2}}e^{is}) |^2 \mid 1-\rho e^{i(\tau+\theta-s)} \mid^{-2} ds.$$

We have

$$\int_0^{2\pi} | I_1 |^2 \tau^{-2\alpha-1} d\tau$$

$$\leqq A \int_{\frac{1}{2}}^1 (1-\rho)^{2+\alpha} d\rho \int_0^{2\pi} | u'(\rho^{\frac{1}{2}}e^{is}) |^2 ds \int_0^{2\pi} | 1-\rho e^{i(\tau+\theta-s)} |^{-2} h_\rho(\tau)\tau^{-\alpha-1} d\tau.$$

It is easily verified that

$$\int_0^{2\pi} | 1-\rho e^{i(\tau-\phi)} | h_\rho(\tau)\tau^{-\alpha-1} d\tau \leqq (1-\rho)^{-1} \mid \phi \mid^{-1-\alpha}, \quad 1-\rho \leqq \mid \phi \mid \leqq \pi,$$

$$\leqq (1-\rho)^{-2-\alpha}, \qquad 0 \leqq \mid \phi \mid \leqq 1-\rho.$$

Thus

$$\int_0^{2\pi} | I_1 |^2 \tau^{-2\alpha-1} d\tau \leqq A \int_{\frac{1}{2}}^1 d\rho \int_{|\theta-s|\leqq 1-\rho} | u'(\rho^{\frac{1}{2}}e^{is}) |^2 ds$$

$$+ A \int_{\frac{1}{2}}^1 (1-\rho)^{1+\alpha} d\rho \int_{|\theta-s|\geqq 1-\rho} | u'(\rho^{\frac{1}{2}}e^{is}) |^2 \mid \theta-s \mid^{-\alpha-1} ds.$$

Setting $\rho^{\frac{1}{2}} = r$, we obtain

$$\int_0^{2\pi} | I_1 |^2 \tau^{-2\alpha-1} d\tau \leqq A \int_0^1 r dr \int_{|\theta-s|\leqq 2(1-r)} | u'(re^{is}) |^2 ds$$

$$+ A \int_0^1 (1-r)^{1+\alpha} dr \int_{|\theta-s|\geqq(1-r)} | u'(re^{is}) |^2 \mid \theta-s \mid^{-\alpha-1} ds$$

$\leqq AG^2{}_{1+\alpha}(\theta) + AS^2(\theta)$. Using Theorems 1.2c and 1.2d, we find that

$$\int_0^{2\pi} [\int_0^{2\pi} | I_1 |^2 \tau^{-2\alpha-1} d\tau]^{\frac{1}{2}p} d\theta \leqq A \parallel u \parallel_p^p.$$

Consider I_2. We have

$$| I_2 | \leqq A\tau^2 \int_\theta^{\theta+\tau} | u'''_\alpha(\rho_\tau e^{it}) | dt, \qquad | I_2 |^2 \leqq A\tau^5 \int_\theta^{\theta+\tau} | u'''_\alpha(\rho_\tau e^{it}) |^2 dt.$$

Making use of (2) with $\rho = \rho_\tau$, we find that

$$|I_2|^2 \leqq A\tau^{2+2\alpha} \int_\theta^{\theta+\tau} dt \int_0^{2\pi} |u'(\rho_\tau^{\frac{1}{2}} e^{is})|^2 \, |1 - \rho_\tau^{\frac{1}{2}} e^{i(t-s)}|^{-2} \, ds$$

$$\leqq A\tau^{2+2\alpha} \int_0^{2\pi} |u'(\rho_\tau^{\frac{1}{2}} e^{is})|^2 \, ds \int_\theta^{\theta+\tau} |1 - \rho_\tau^{\frac{1}{2}} e^{i(t-s)}|^{-2} \, dt.$$

Since $\int_\theta^{\theta+\tau} |1 - \rho_\tau^{\frac{1}{2}} e^{i(t-s)}|^{-2} \, dt \leqq AP(\rho_\tau^{\frac{1}{2}}, \theta - s)$, we have

$$|I_2|^2 \leqq A\tau^{2+2\alpha} \int_0^{2\pi} |u'(\rho_\tau^{\frac{1}{2}} e^{is})|^2 \, P(\rho_\tau^{\frac{1}{2}}, \theta - s) ds,$$

$$\int_0^{2\pi} \tau^{-2\alpha-1} |I_2|^2 d\tau \leqq A \int_0^{2\pi} d\tau \int_0^{2\pi} |u'(\rho_\tau^{\frac{1}{2}} e^{is})|^2 \, P(\rho_\tau^{\frac{1}{2}}, \theta - s) ds$$

$$\leqq A \int_{\frac{1}{2}}^1 (1-\rho) d\rho \int_0^{2\pi} |u'(\rho^{\frac{1}{2}}, e^{is})|^2 \, P(\rho^{\frac{1}{2}}, \theta - s) ds$$

$$\leqq A \int_0^1 (1-r) dr \int_0^{2\pi} |u'(re^{is})|^2 \, P(r, \theta - s) ds.$$

From this it follows, as before, that

$$\int_0^{2\pi} \Big[\int_0^{2\pi} \tau^{-2\alpha-1} |I_2|^2 d\tau \Big]^{\frac{1}{2}p} d\tau \leqq A \| u \|_p^p.$$

Consider I_3. We have $|I_3| \leqq A |\tau|^2 |u''_\alpha(\rho_\tau e^{i\theta})|$. Since

$$u''_\alpha(\rho e^{i\theta}) = (\rho e^{i\theta})^{-2} [-u_{\alpha-2}(\rho e^{i\theta}) + iu_{\alpha-1}(\rho e^{i\theta})],$$

it follows that

$$\int_0^{2\pi} \tau^{-2\alpha-1} |I_3|^2 d\tau \leqq A \int_{\frac{1}{2}}^1 (1-\rho)^{3-2\alpha} |u_{\alpha-2}(\rho e^{i\theta})|^2 \, d\rho$$

$$+ A \int_{\frac{1}{2}}^1 (1-\rho)^{3-2\alpha} |u_{\alpha-1}(\rho e^{i\theta})|^2 \, d\rho$$

$$\leqq A \int_0^1 (1-\rho)^{3-2\alpha} |u_{\alpha-2}(\rho e^{i\theta})|^2 \, d\rho$$

$$+ A \int_0^1 (1-\rho)^{1-2\alpha} |u_{\alpha-1}(\rho e^{i\theta})|^2 \, d\rho$$

$$\leqq A\Delta(\alpha - 1, u, \theta)^2 + A\Delta(\alpha, u, \theta)^2.$$

By Theorem 2.1, $\int_0^{2\pi} \Big[\int_0^{2\pi} \tau^{-2\alpha-1} |I_3|^2 \, d\tau \Big]^{\frac{1}{2}p} d\theta \leqq A \| u \|_p^p$.

Similar arguments serve to show that

$$\int_0^{2\pi} \Big[\int_0^{2\pi} \tau^{-2\alpha-1} |I_k|^2 d\tau \Big]^{\frac{1}{2}p} d\theta \leqq A \| u \|_p^p, \qquad k = 4, 5, 6.$$

Combining these six inequalities, we obtain our theorem.

3.2. We shall here prove the converse inequality.

THEOREM 3.2. *If* $1 < p < \infty$, $0 < \alpha < 1$, *then* $\| \delta(\alpha, u, \theta) \|_p \geqq A \| u \|_p$.

We will show that

$$\delta(\alpha, u, \theta) > A\Delta(\alpha, u, \theta). \tag{1}$$

Our result will then follow from Theorem 2.2. We have

$$u_\alpha(\rho, \theta) = \int_0^{2\pi} u_\alpha(t) P(\rho, \theta - t)\, dt.$$

Differentiating with respect to θ,

$$u_{\alpha-1}(\rho, \theta) = \int_0^{2\pi} u_\alpha(t) P_\theta(\rho, \theta - t)\, dt.$$

Since P_θ is odd, we find that

$$u_{\alpha-1}(\rho, \theta) = \tfrac{1}{2} \int_0^{2\pi} [u_\alpha(\theta - \tau) - u_\alpha(\theta + \tau)] P_\theta(\rho, \tau)\, d\tau.$$

Now $| P_\theta(\rho, \tau) | \leqq A | 1 - \rho e^{i\tau} |^{-2}$. By Schwarz's inequality

$$| u_{\alpha-1}(\rho, \theta)| \leqq A \int_0^{2\pi} | u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau)| \, | 1 - \rho e^{i\tau} |^{-2}\, d\tau,$$

$$| u_{\alpha-1}(\rho, \theta)|^2 \leqq A \int_0^{2\pi} | u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau)|^2 \, | 1 - \rho e^{i\tau} |^{-2-\alpha}\, d\tau \times \int_0^{2\pi} | 1 - \rho e^{i\tau} |^{-2+\alpha}\, d\tau$$

$$\leqq A(1 - \rho)^{-1+\alpha} \int_0^{2\pi} | u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau)|^2 \, | 1 - \rho e^{i\tau} |^{-2-\alpha}\, d\tau.$$

Thus

$$\int_0^1 (1 - \rho)^{1-2\alpha} \, | u_{\alpha-1}(\rho, \theta)|^2 \, d\rho \leqq A \int_0^1 (1 - \rho)^{-\alpha}\, d\rho \int_0^{2\pi} | u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau)|^2 \, | 1 - \rho e^{i\tau} |^{-2-\alpha}\, d\tau,$$

$$\leqq A \int_0^{2\pi} | u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau)|^2\, d\tau \int_0^1 (1 - \rho)^{-\alpha} \, | 1 - \rho e^{i\tau} |^{-2-\alpha}\, d\rho.$$

Now $\int_0^1 (1 - \rho)^{-\alpha} \, | 1 - \rho e^{i\tau} |^{-2-\alpha}\, d\rho \leqq A\tau^{-1-2\alpha}$, and thus

$$\int_0^1 (1 - \rho)^{1-2\alpha} \, | u_{\alpha-1}(\rho, \theta)|^2 \, d\rho \leqq A \int_0^{2\pi} | u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau)|^2 \, \tau^{-2\alpha-1}\, d\tau.$$

Theorems 3. 1 and 3. 2 are false if $\alpha = 1$. However, they can be modified so that they become true. Marcenkiewicz [6] conjectured and Zygmund [8] proved that

$$A \| u \|_p^p \leqq \int_0^{2\pi} [\int_0^{2\pi} | u_1(\theta + \tau) - 2u_1(\theta) + u_1(\theta - \tau)|^2 \tau^{-3} \, d\tau]^{\frac{1}{2}p} \, d\theta$$
$$\leqq A \| u \|_p^p.$$

4. 1. Let $u(\theta) \in L_1(0, 2\pi)$ and have mean value zero. Let $u_{\alpha-1}(\rho, \theta)$ be defined as in § 1. 1 and let $u\dagger(\theta)$ be defined as in § 1. 2.

LEMMA 4. 1. $\quad | u_{\alpha-1}(\rho, \theta)| \leqq A(1-\rho)^{-1+\alpha} u\dagger(\theta).$

We have $u_{\alpha-1}(\rho, \theta) = \int_{-\pi}^{\pi} u(\theta - t) k(\alpha, \rho, t) \, dt$, where

$$k(\alpha, \rho, t) = \tfrac{1}{2\pi} \Sigma' e^{int} \rho^{|n|} / (in)^{\alpha-1}.$$

Let $U(\theta, t) = \int_0^t u(\theta - \tau) \, d\tau$. We have $U(\theta, t) \leqq | t | \, u\dagger(\theta)$. Integrating by parts we obtain, since the integrated term vanishes,

$$u_{\alpha-1}(\rho, \theta) = -\int_{-\pi}^{\pi} U(\theta, t) \partial k(\alpha, \rho, t)/\partial t \, dt,$$

$$| u_{\alpha-1}(\rho, \theta)| \leqq u\dagger(\theta) \int_{-\pi}^{\pi} | \, t \, \partial k(\alpha, \rho, t)/\partial t \, | \, dt.$$

Now $\partial k(\alpha, \theta, t)/\partial t \, | \leqq A \, | \, 1 - \rho e^{it} \, |^{-3+\alpha}$, from which it follows that

$$\int_{-\pi}^{\pi} | \, t \, \partial k(\alpha, \rho, t)/\partial t \, | \, dt \leqq A(1-\rho)^{-1+\alpha}$$

Inserting this, our lemma is proved.

Let $1 < r < \infty$. We define

$$\Delta(r, \alpha, u, \theta) = [\int_0^1 | \, u_{\alpha-1}(\rho, \theta)|^r \, (1-\rho)^{r-1-\alpha r} \, d\rho]^{1/r}.$$

THEOREM 4. 1. *If* $1 < p < \infty$, $-\infty < \alpha < 1$, *then*

$$\| \Delta(r, \alpha, u, \theta)\|_p \leqq A \| u \|_p \text{ for } 2 \leqq r < \infty;$$

$$\| \Delta(r, \alpha, u, \theta)\|_p \geqq A \| u \|_p \text{ for } 1 < r \leqq 2.$$

Suppose first that $r \geqq 2$. Then, by Lemma 4. 1,

$$\Delta(r,\alpha,u,\theta)^r = \int_0^1 |u_{\alpha-1}(\rho,\theta)|^r\,(1-\rho)^{r-1-\alpha r}\,d\rho$$

$$\leqq A\Big[\int_0^1 |u_{\alpha-1}(\rho,\theta)|^2\,(1-\rho)^{1-2\alpha}d\rho\Big][u\dagger(\theta)]^{r-2},$$

so that $\Delta(r,\alpha,u,\theta) \leqq A\Delta(2,\alpha,u,\theta)^{2/r}[u\dagger(\theta)]^{1-2/r}$. Using Hölder's inequality and Theorem 2. 1, we obtain

$$\|\Delta(r,\alpha,u,\theta)\|_p \leqq A\,\|\Delta(2,\alpha,u,\theta)\|_p^{2/r}\,\|u\dagger(\theta)\|_p^{1-2/r} \leqq A\,\|u\|_p^p.$$

Next let $1 < r \leqq 2$. We have

$$\Delta(2,\alpha,u,\theta)^2 = \int_0^1 |u_{\alpha-1}(\rho,\theta)|^2\,(1-\rho)^{1-2\alpha}\,d\rho$$

$$\leqq A\Big[\int_0^1 |u_{\alpha-1}(\rho,\theta)|^r\,(1-\rho)^{r-1-\alpha r}d\rho\Big][u\dagger(\theta)]^{2-r},$$

so that $\Delta(2,\alpha,u,\theta) \leqq A\Delta(r,\alpha,u,\theta)^{\frac{1}{2}r}[u\dagger(\theta)]^{1-\frac{1}{2}r}$. Using Hölder's inequality and Theorem 2. 2, we obtain

$$\|u\|_p \leqq A\,\|\Delta(r,\alpha,u,\theta)\|_p^{\frac{1}{2}r}\,\|u\|_p^{1-\frac{1}{2}r}, \qquad \|\Delta(r,\alpha,u,\theta)\|_p \geqq A\,\|u\|_p.$$

For similar results see [7].

4. 2. The two lemmas which follow are analogous to results proved in Hardy and Littlewood [2].

LEMMA 4. 2a. *If* $u(\theta) \in L^p(0,2\pi)$, $1 < p < \infty$, $0 < \alpha < 1$, *and* $1/p + 1/q = 1$, *then*

$$\int_0^{2\pi} \|u(\theta+\tau) - u(\theta-\tau)\|_p^p\,\tau^{-1-p\alpha}\,d\tau$$

$$\leqq A\int_0^1 \|u_{-1}(\rho,\theta)\|_p^p\,(1-\rho)^{p(1/q-\alpha)}\,d\rho.$$

Let $\rho_\tau = 1 - (\tau/4\pi)$. We have $u(\theta+\tau) - u(\theta-\tau) = I_1 + I_2 + I_3$, where

$$I_1 = \int_{\rho_\tau}^1 \partial u(\rho,\theta+\tau)/\partial\rho\,d\rho, \qquad I_2 = \int_{-\tau}^{\tau} \partial u(\rho_\tau,\theta+\phi)/\partial\theta\,d\phi,$$

$$I_3 = \int_1^{\rho_\tau} \partial u(\rho,\theta-\tau)/\partial\rho\,d\rho.$$

By Minkowski's inequality,

$$\| I_2 \|_p = \Big(\int_0^{2\pi} d\theta \, \Big| \int_\tau^\tau \partial u(\rho_\tau, \theta + \phi)/\partial\theta d\phi \, \Big|^p \Big)^{1/p}$$

$$\leqq \int_{-\tau}^{\tau} d\phi \Big(\int_0^{2\pi} | \, u_{-1}(\rho_\tau, \theta + \phi) \, |^p \, d\theta \Big)^{1/p}$$

$\leqq A \, | \, \tau \, | \, \| \, u_{-1}(\rho_\tau, \theta) \, \|_p$. Thus

$$(1) \qquad \int_0^{2\pi} \| \, I_2 \, \|_p^p \tau^{-1-p\alpha} \, d\tau \leqq A \int_{\frac{1}{2}}^1 \| \, u_{-1}(\rho, \theta) \, \|_p^p (1 - \rho)^{p-1-p\alpha} \, d\rho$$

$\leqq A \int_0^1 \| \, u_{-1}(\rho, \theta) \, \|_p^p (1 - \rho)^{p-1-p\alpha} \, d\rho$. Consider I_2. Since $\partial u/\partial\rho = u^{\infty}_{-1}/\rho$,

$$\| \, I_1 \, \|_p = \Big(\int_0^{2\pi} d\theta \, \Big| \int_{\rho_\tau}^1 1/\rho u^{\infty}_{-1}(\rho, \theta + \tau) \, d\rho \, \Big|^p \Big)^{1/p}$$

$$\leqq \int_{\rho_\tau}^1 1/\rho \, \| \, u^{\infty}_{-1}(\rho, \theta) \, \|_p \, d\rho \leqq A \int_{\rho_\tau}^1 \| \, u_{-1}(\rho, \theta) \, \|_p \, d\rho.$$

Here we have used the fact that $\rho \geqq \frac{1}{2}$ for all τ and M. Riesz's theorem on conjugate functions. Let λ be a parameter such that $q\lambda < 1$, $p\lambda > -1$. We have

$$\| \, I_1 \, \|_p \leqq \int_{\rho_\tau}^1 \| \, u_{-1}(\rho, \theta) \, \|_p \, (1 - \rho)^{\lambda} (1 - \rho)^{-\lambda} \, d\rho.$$

By Hölder's inequality,

$$\| \, I_1 \, \|_p^p \leqq \int_{\rho_\tau}^1 \| \, u_{-1}(\rho, \theta) \, \|_p^p \, (1 - \rho)^{p\lambda} \, d\rho \Big(\int_{\rho_\tau}^1 (1 - \rho)^{-q\lambda} \, d\rho \Big)^{p/q}.$$

We have

$$\int_{\rho_\tau}^1 (1 - \rho)^{-q\lambda} d\rho \leqq A (1 - \rho_\tau)^{1-q\lambda},$$

$$\| \, I_1 \, \|_p^p \, \tau^{-\alpha p-1} \leqq A (1 - \rho_\tau)^{-\lambda p - \alpha p - 1 + p/q} \int_{\rho_\tau}^1 \| \, u_{-1}(\rho, \theta) \, \|_p^p \, (1 - \rho)^{p\lambda} \, d\rho,$$

$$\int_0^{2\pi} \| \, I_1 \, \|_p^p \, \tau^{-\alpha p-1} \, d\tau$$

$$\leqq A \int_0^{2\pi} (1 - \rho_\tau)^{-\lambda p - \alpha p - 1 + p/q} \, d\tau \int_{\rho_\tau}^1 \| \, u_{-1}(\rho, \theta) \, \|_p^p \, (1 - \rho)^{p\lambda} \, d\rho$$

$$\leqq A \int_{\frac{1}{2}}^1 \| \, u_{-1}(\rho, \theta) \, \|_p^p \, (1 - \rho)^{p\lambda} \, d\rho \int_{4\pi(1-\rho)}^{2\pi} (1 - \rho_\tau)^{-\lambda p - \alpha p - 1 + p/q} \, d\tau.$$

If $-\lambda p - \alpha p + p/q < 0$, which for $\alpha > 0$ can be effected by a suitable choice of λ, then

$$\int_{4\pi(1-\rho)}^{2\pi}(1-\rho_\tau)^{-\lambda p-\alpha p-1+p/q}d\tau \leqq A(1-\rho)^{-\lambda p-\alpha p+p/q},$$

$$\int_0^{2\pi}\| I_1 \|_p^p \tau^{-\alpha p-1}d\tau \leqq A\int_0^1 \| u_{-1}(\rho,\theta)\|_p^p (1-\rho)^{-\alpha p+p/q}\,d\rho. \tag{2}$$

In an exactly similar way we can show that

$$\int_0^{2\pi}\| I_3 \|_p^p \tau^{-\alpha p-1}\,d\tau \leqq A\int_0^1 \| u_{-1}(\rho,\theta)\|_p^p (1-\rho)^{-\alpha p+p/q}\,d\rho. \tag{3}$$

By Jensen's inequality,

$$\| u(\theta+\tau)-u(\theta-\tau)\|_p^p \leqq 3^{p-1}(\| I_1\|_p^p+\| I_2\|_p^p+\| I_3\|_p^p)$$

and thus, using (1), (2) and (3),

$$\int_0^{2\pi}\| u(\theta+\tau)-u(\theta-\tau)\|_p^p \tau^{-1-\alpha p}\,d\tau \leqq A\int_0^1 \| u_{-1}(\rho,\theta)\|_p^p (1-\rho)^{-\alpha p+p/q}\,d\rho.$$

LEMMA 4.2b. *If* $1<p<\infty$, $-1/p<\alpha<1$, $1/p+1/q=1$ *if*

$$u(\theta)\,\varepsilon\, L^p(0,2\pi),$$

then

$$\int_0^1 \| u_{-1}(\rho,\theta)\|_p^p (1-\rho)^{p(1/q-\alpha)}\,d\rho \leqq A\int_0^1 \| u(\theta+\tau)-u(\theta-\tau)\|_p^p\, \tau^{-1-\alpha p}\,d\tau.$$

We have

$$u_{-1}(\rho,\theta)=\tfrac{1}{2}\int_0^{2\pi}[u(\theta-\tau)-u(\theta+\tau)]\,\partial P(\rho,\tau)/\partial\tau\, d\tau.$$

Since $|\partial/P(\rho,\tau)\partial\tau| \leqq A\,|\rho e^{i\tau}-1|^{-2}$, we have, using Hölder's inequality,

$$|u_{-1}(\rho,\theta)|^p \leqq A\int_0^{2\pi}|u(\theta+\tau)-u(\theta-\tau)|^p\,|\rho e^{i\tau}-1|^{-p\lambda}d\tau\Big[\int_0^{2\pi}|\rho e^{i\tau}-1|^{-q\mu}d\tau\Big]^{p/q}.$$

Here λ and μ are parameters such that $\lambda+\mu=2$, $q\mu>1$. Because of this last condition $\int_0^{2\pi}|\rho e^{i\tau}-1|^{-q\mu}\,d\tau \leqq A(1-\rho)^{1-q\mu}$, and thus

$$|u_{-1}(\rho,\theta)|^p \leqq A(1-\rho)^{-p\mu+p/q}\int_0^{2\pi}|u(\theta+\tau)-u(\theta-\tau)|^p\,|e^{i\tau}\rho-1|^{-p\lambda}\,d\tau,$$

$$\| u_{-1}(\rho,\theta)\|_p^p \leqq A(1-\rho)^{-p\mu+p/q}\int_0^{2\pi}\| u(\theta+\tau)-u(\theta-\tau)\|_p^p\,|e^{i\tau}\rho-1|^{-p\lambda}d\tau,$$

$$\int_0^1 \| u_{-1}(\rho,\theta)\|_p^p (1-\rho)^{-p\alpha+p/q}d\rho$$

$$\leqq A \int_0^{2\pi} \| u(\theta+\tau) - u(\theta-\tau) \|_p^p \, d\tau \int_0^1 (1-\rho)^{-p\alpha-p\mu+2p/q} \mid e^{i\tau}\rho - 1 \mid^{-p\lambda} d\rho.$$

If $-p\alpha - p\mu + 2p/q > -1$, $p\lambda + p\mu + p\alpha - 2p/q > 1$ then

$$\int_0^1 (1-\rho)^{-p\alpha-p\mu+2p/q} \mid e^{i\tau}\rho - 1 \mid^{-p\lambda} d\rho \leqq A \mid \tau \mid^{-1-p\alpha}$$

and

$$\int_0^1 \| u_{-1}(\rho, \theta) \|_p^p \, (1-\rho)^{-p\alpha+p/q} d\rho$$

$$\leqq A \int_0^{2\pi} \| u(\theta+\tau) - u(\theta-\tau) \|_p^p \tau^{-1-p\alpha} d\tau.$$

If $-1/p < \alpha < 1$, then the conditions on the parameters can be satisfied.

Using these we can prove a result which is analogous to Theorem 4. 1 for the special choice of the parameter r, $r = p$.

THEOREM 4. 2a. *If* $0 < \alpha < 1$, *then*

$$\int_0^{2\pi} \| u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) \|_p^p \tau^{-1-p\alpha} d\tau \geqq A \| u \|_p^p, \qquad 1 < p \leqq 2;$$

$$\int_0^{2\pi} \| u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) \|_q^q \tau^{-1-q\alpha} d\tau \leqq A \| u \|_q^q, \qquad 2 \leqq q < \infty.$$

This is a consequence of Lemmas 4. 2a and 4. 2b and Theorem 4. 1. The following result complements Theorem 4. 2a.

THEOREM 4. 2b. *If* $0 < \alpha < 1$, *then*

$$\int_0^{2\pi} \| u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) \|_p^2 \tau^{-1-2\alpha} d\tau \geqq A \| u \|_p^2, \qquad 1 < p \leqq 2;$$

$$\int_0^{2\pi} \| u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) \|_q^2 \tau^{-1-2\alpha} d\tau \leqq A \| u \|_q^2, \qquad 2 \leqq q < \infty.$$

Minkowski's inequality gives

$$\int_0^{2\pi} \| u_\alpha(\theta+\tau) - u_\alpha(\theta-\tau) \|_r^2 \tau^{1-2\alpha} d\tau \geqq [\int_0^{2\pi} \delta(\alpha, u, \theta)^r d\theta]^{2/r} \quad (r \geqq 2),$$

$$\leqq [\int_0^{2\pi} \delta(\alpha, u, \theta)^r d\theta]^{2/r} \quad (r \leqq 2).$$

Appealing to Theorems 3. 1 and 3. 2 we obtain our desired result.

The result given in the introduction is an immediate corollary of these two theorems combined with the relation

$$\| u_\alpha(\theta + \tau) - u_\alpha(\theta - \tau) \|_p \leqq A \| u \|_p | \tau |^\alpha.$$

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A CLASS OF d-SIMPLE SEMIGROUPS.*

By A. H. CLIFFORD.

In a recent paper, J. A. Green [2] proposed the determination of all regular d-simple semigroups (definitions in § 1 below) as the next step beyond the determination by Suschkewitsch [4] and Rees [3] of all completely simple semigroups. The present paper takes a small portion of this step, dealing with semigroups S satisfying the following conditions:

A1. *S is d-simple.*

A2. *S has an identity element.*

A3. *Any two idempotent elements of S commute.*

(Regularity is a consequence of A1 and A2.) It is shown that the structure of S is determined by that of its right unit subsemigroup P, and that P has the following properties:

B1. *The right cancellation law holds in P.*

B2. *P has an identity element.*

B3. *The intersection of two principal left ideals of P is a principal left ideal.*

Conversely, if P is any semigroup having properties B1, 2, 3, there exists a semigroup S satisfying A1, 2, 3, the right unit subsemigroup of which is isomorphic with P. A detailed statement of this result is given in § 1, and the rest of the paper is devoted to the proof thereof.

The problem of describing all semigroups S satisfying A1, 2, 3 is of course merely replaced by the perhaps equally difficult one of describing all semigroups P satisfying B1, 2, 3. The extent of the class of all such P can be seen from the fact that the positive part of a lattice-ordered group ([1], Chapter 14) satisfies not only B1, 2, 3 but also the left-right duals thereof.

The present work parallels rather than extends that of Suschkewitsch and Rees. For while a completely simple semigroup is regular and d-simple, it reduces to a group if either of the conditions A2 or A3 is imposed upon it.

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1. Definitions and statement of main result. Let S be any semigroup with identity element 1. We say that two elements a and b of S are right associate if they generate the same principal right ideal: $aS = bS$. An element p of S is right associate to 1 if and only if it has a right inverse q in S: $pq = 1$. Such an element p will be called a right unit of S. The set P of all right units of S is a subsemigroup of S which we shall call the right unit subsemigroup of S. P evidently satisfies B1 and B2. Left associate elements, left units, and the left unit subsemigroup Q of S, are defined analogously. Two elements of S will be called associate if they are both left and right associate. The intersection $U = P \cap Q$ of P and Q consists of all elements associate to 1, i. e. of all elements having inverses on both sides. U is evidently a subgroup of S. Its elements will be called units.

Two elements a and b of S are called d-equivalent (Green [2], p. 164) if there exists an element S which is left associate to a and right associate to b. (This implies the existence of an element of S which is right associate to a and left associate to b.) We shall say that S is d-simple if it consists of a single class of d-equivalent elements. A d-simple semigroup is simple, but not conversely.

Now let P be any semigroup satisfying B1, 2, 3. From each class of left associate elements of P let us pick a fixed representative. B3 states that if a and b are elements of P, there exists c in P such that $Pa \cap Pb = Pc$. c is determined by a and b only to within left associates. We define $a \vee b$ to be the representative of the class to which c belongs. $a \vee b$ evidently has the properties of a least common left multiple (LCLM) of a and b. We observe also that $a \vee b = b \vee a$. We define a binary operation $*$ by

$$(a * b)b = a \vee b \tag{1.1}$$

for each pair of elements a, b of P. There is at least one such element in P since $a \vee b$ is in Pb, and at most one by B1.

Now let $P^{-1} \circ P$ denote the set of ordered pairs (a, b) of elements of P with equality defined by

$$(a, b) = (a', b') \text{ if } a' = ua, b' = ub \tag{1.2}$$

where u is a unit in P. We define product in $P^{-1} \circ P$ by

$$(a, b)(c, d) = ((c * b)a, (b * c)d). \tag{1.3}$$

Main Theorem. *Starting with a semigroup P satisfying* B1, 2, 3, *equations* (1.1), (1.2), (1.3) *define a semigroup $P^{-1} \circ P$ satisfying* A1, 2, 3; *P is isomorphic with the right unit subsemigroup of $P^{-1} \circ P$. Conversely, if*

S is a semigroup satisfying A1, 2, 3, *its right unit subsemigroup P satisfies* B1, 2, 3 *and S is isomorphic with $P^{-1} \circ P$.*

Of course we shall have to show that (1. 3) is single-valued, i. e. consistent with (1. 2), and that the associative law holds, in addition to the properties mentioned. But let us point out here that *the definition* (1. 3) *is independent of the choice of representative elements in the classes of left associates in P.* For a new choice would (by Lemma 4. 1) replace each $a \vee b$ by $u_{a,b}\ (a \vee b)$ with $u_{a,b}$ a unit. From $(a * b)b = a \vee b = b \vee a = (b * a)a$ we see that $a * b$ and $b * a$ are multiplied on the left by the same $u_{a,b}$. By (1. 2), this has no effect on the product as defined by (1. 3).

Let us also mention here without proof that, in the case $U = \{1\}$, $P^{-1} \circ P$ can be isomorphically represented by the semigroup of mappings of P into itself generated by the mappings p_a, q_a (a an arbitrary element of P) defined as follows: $q_a x = xa$, $p_a x = x * a$ (x a variable element of P). One shows that $q_a q_b = q_{ba}$, $p_a p_b = p_{ab}$, and $p_a q_b = q_{b*a} p_{a*b}$. This construction avoids the troublesome proof of associativity, but does not seem applicable in the presence of units $\neq 1$.

2. d-simple semigroups with identity element. In this section we deal with semigroups S satisfying A1 and A2, but not necessarily A3.

THEOREM 2. 1. *Let S be a semigroup with identity element, and let P and Q be its right and left unit subsemigroups. Then S is d-simple if and only if $S = QP$.*

Proof. Assume first that S is d-simple, and let a be an arbitrary element of S. Since a is d-equivalent to 1, there exists an element p of S which is left associate to a and right associate to 1. The latter means that p is in P. The former means that elements x and y exist in S such that $a = xp$ and $p = ya$. Let q be a right inverse of p. Then $1 = pq = yaq = yxpq = yx$. Hence x is in Q, and from $a = xp$ we infer $S = QP$.

Suppose conversely that $S = QP$. Let $a = qp (p \in P, q \in Q)$ be any element of S. By definition of P and Q there exist x and y in S such that $px = yq = 1$. From $a = qp$ and $p = ya$ we see that a and p are left associate. Since 1 and p are right associate, a is d-equivalent to 1, and S is d-simple.

A semigroup S is said to be regular if to each a in S there exists an element x of S such that $axa = a$ (Green [2], p. 163).

COROLLARY. *A d-simple semigroup S with identity element is regular.*

Proof. Let $a = qp (p \in P, q \in Q)$ be any element of S. There exist y and z in S such that $py = zq = 1$. Let $x = yz$. Then

$$axa = qpyzqp = q \cdot 1 \cdot 1 \cdot p = qp = a.$$

In Lemmas 2. 1-2. 4, S will denote a semigroup with identity element, P and Q its right and left unit subsemigroups, and $U = P \cap Q$ its group of units. Lemmas 2. 2-2. 4 are concerned with properties of the complex QP. Our ultimate interest is, of course, in the case $QP = S$, but the validity of these lemmas does not depend on this assumption.

LEMMA 2. 1. *The group of units of P is the group U of units of S. For any element p of P, $Pp = P \cap Sp$, i. e. an element p' of P is a left multiple of p in S if and only if it is a left multiple of p in P. Two elements of P are associate in S if and only if they are left associate in P. This in turn holds if and only if the two elements differ by a unit factor in the left; this unit factor is unique.*

Proof. $U = P \cap Q$, where Q is the left unit subsemigroup of S, and hence U is contained in the group of units of P. But a unit in P has a left inverse in P, hence also in S, hence belongs to Q, and therefore to $P \cap Q = U$. If p' is a left multiple of p in S, say $p' = xp$, and if q' is a right inverse of p' in S, then $1 = p'q' = x \cdot pq'$. Hence $x \in P$, and p' is a left multiple of p in P.

Consequently, two elements p and p' of P are left associate in P if and only if they are left associate in S, hence if and only if they are associate in S, since any two elements of P are right associate in S. If this is the case, then $p = xp'$ and $p' = yp$ with x and y in P. From $p = xyp$, $p' = yxp'$, and right cancellation in P, we infer $xy = yx = 1$, i. e. x and y are units. The unicity of x and y also follow from right cancellation in P.

The left-right dual of Lemma 2. 1 also holds, the proof being similar.

LEMMA 2. 2. *An element $e = qp$ of QP is idempotent if and only if $pq = 1$.*

Proof. If $pq = 1$, $e^2 = qpqp = q1p = qp = e$. Suppose conversely that $e = qp$ is idempotent. Since $p \in P$ and $q \in Q$, elements x and y exist in S such that $px = yq = 1$. Then $pq = yq \cdot pq \cdot px = y(qp)^2x = yqpx = 1 \cdot 1 = 1$.

LEMMA 2. 3. *If $a = qp$ is any element of QP, then p [q] is left [right] associate to a. If $a = q'p'$ is any other expression of a as the product of an element q' of Q by an element p' of P, then there exists a unit u such that $p' = up$, $q' = qu^{-1}$.*

Proof. Let x [y] be a left [right] inverse of q [p]: $xq = py = 1$. From $a = qp$ and $p = xa$ we infer $Sa = Sp$. From $a = qp$ and $q = ay$ we infer $aS = qS$. If $a = q'p'$ then $Sp = Sa = Sp'$ and $qS = aS = q'S$. From Lemma 2.1 there exist units u and v such that $p' = up$, $q' = qv$. Then

$$1 = xqpy = xay = xq'p'y = xqvupy = vu.$$

Hence $p' = up$, $q' = qu^{-1}$.

LEMMA 2.4. *If $a = qp$ is any element of QP, then any element of QP associate to a is expressible in the form qup with u a unit, and u is unique.*

Proof. Let $b = q'p'$ be an element of QP associate to a. Then, by Lemma 2.3, $Sp' = Sb = Sa = Sp$, and, by Lemma 2.1, $p' = vp$ with v a unit. Similarly $q' = qw$ with w a unit. Then $u = wv$ is a unit, and $b = q'p' = qwvp = qup$. Suppose $b = qup = qu'p$ with units u and u'. There exist elements x, y of S such that $xq = py = 1$. Then

$$u = xq \cdot u \cdot py = xby = xq \cdot u' \cdot py = u'.$$

THEOREM 2.2. *The following three conditions on a d-simple semigroup S with identity element are equivalent.*

A3. *The idempotent elements of S commute.*

A4. *Every principal left ideal of S, and every principal right ideal, has a unique idempotent generator.*

A5. *Every right unit of S has a unique right inverse, and every left unit a unique left inverse in S.*

Proof. I. A3 implies A4. That every principal left or right ideal of S has at least one idempotent generator follows from the regularity of S (Corollary to Theorem 2.1). To show the unicity, suppose $eS = fS$ ($e^2 = e$, $f^2 = f$). Then $e = fx$ and $f = ey$ for some x and y in S, whence $fe = e$ and $ef = f$. A3 then implies $e = f$.

II. A4 implies A5. Suppose q and q' are right inverses of the element p of P: $pq = pq' = 1$. Then, by Lemmas 2.2 and 2.3, $e = qp$ and $e' = q'p$ are idempotent elements generating the same principal left ideal Sp. By A4, $e = e'$. Then $q = qpq = eq = e'q = q'pq = q'$.

III. A5 implies A3. Let e_1 and e_2 be any two idempotent elements of S. Since S is d-simple, $S = QP$ by Theorem 2.1, and hence $e_1 = q_1p_1$ and $e_2 = q_2p_2$ with q_1 and q_2 in Q, p_1 and p_2 in P. By Lemma 2.2, $p_1q_1 = p_2q_2 = 1$. Again using $S = QP$, $p_1q_2 = q_3p_4$ and $p_2q_1 = q_5p_6$ with q_3 and q_5 in Q,

p_4 and p_6 in P. Let p_3 and p_5 be respective left inverses of q_3 and q_5; likewise q_4 and q_6 right inverses of p_4 and p_6. Then $p_3p_1q_2q_4 = p_3q_3p_4q_4 = 1$, while $p_3p_1q_1q_3 = 1$ and $p_4p_2q_2q_4 = 1$. By A5 we infer that $q_1q_3 = q_2q_4$ and $p_3p_1 = p_4p_2$. Similarly, from $p_5p_2q_1q_6 = p_5q_5p_6q_6 = 1$, $p_5p_2q_2q_5 = 1$, and $p_6p_1q_1q_6 = 1$, we infer that $q_1q_6 = q_2q_5$ and $p_6p_1 = p_5p_2$. Then

$$p_4q_5 \cdot p_6q_3 = p_4p_2q_1q_3 = p_3p_1q_1q_3 = 1 \quad \text{and} \quad p_6q_3 \cdot p_4q_5 = p_6p_1q_2q_5 = p_6p_1q_1q_6 = 1.$$

Hence $p_4q_5 = u$ and $p_6q_3 = u^{-1}$ with u a unit. From $p_4 \cdot q_5u^{-1} = 1$ and $up_6 \cdot q_3 = 1$ and A5 we conclude that $q_5u^{-1} = q_4$ and $up_6 = p_3$. Finally,

$$e_1e_2 = q_1p_1q_2p_2 = q_1q_3p_4p_2 = q_2q_4p_3p_1 = q_2q_5u^{-1}up_6p_1$$
$$= q_2q_5p_6p_1 = q_2p_2q_1p_1 = e_2e_1.$$

3. Introduction of condition A3. We now assume that S is a semigroup satisfying A1, 2, 3. Then S has the properties A4 and A5 of Theorem 2. 2. By A5, the relation $pq = 1$ defines a one-to-one correspondence $p \leftrightarrow q$ between P and Q which is evidently an anti-isomorphism.

We shall now adopt a new notation. Elements of P will be denoted by $a, b, c, \cdots$, and their corresponding right inverses in Q by $a^{-1}, b^{-1}, c^{-1}, \cdots$. Thus we have the rules $aa^{-1} = 1$, $(ab)^{-1} = b^{-1}a^{-1}$. By Theorem 2. 1, every element of S is expressible in the form $a^{-1}b$ with a and b in P. By Lemma 2. 2, $a^{-1}a$ is an idempotent element of S which we shall denote by e_a. Thus $a^{-1}a = e_a$, $ae_a = a$, $e_aa^{-1} = a^{-1}$. Elements of U will, as before, be denoted by $u, v, w, \cdots$. $e_a = 1$ if and only if $a \in U$.

LEMMA 3. 1. *$e_a = e_b$ if and only if $Pa = Pb$ $(a, b \in P)$.*

Proof. By Lemma 2. 1, $Pa = Pb$ if and only if $Sa = Sb$. Since $Sa = Se_a$ and $Sb = Se_b$, $Pa = Pb$ if and only if $Se_a = Se_b$, and by A4 (Theorem 2. 2), the latter holds if and only if $e_a = e_b$.

By A3, the product of two idempotents is idempotent. The set E of idempotents of S is thus a commutative subsemigroup of S, and hence a semilattice (cf. [1], Example 1, p. 18 and Example 4, p. 25) under the usual definition $e \leqq f$ if $ef = fe = e$. By Lemma 2. 2, every idempotent of S has the form e_a with $a \in P$. For given $a, b \in P$, e_ae_b is idempotent, and hence there exists an element $c \in P$ such that e_ae_b $(= e_be_a) = e_c$.

LEMMA 3. 2. *If $e_ae_b = e_be_a = e_c$ $(a, b, c \in P)$, then $Pa \cap Pb = Pc$, and conversely.*

Proof. Suppose $e_a e_b = e_b e_a = e_c$. Since $S = QP$ (Theorem 2.1), there exist a_1 and b_1 in P such that $ab^{-1} = b_1^{-1}a_1$. From

$$c^{-1}c = a^{-1}a \cdot b^{-1}b = a^{-1}b_1^{-1}a_1 b = (b_1 a)^{-1}(a_1 b)$$

and Lemma 2.3 we infer that c is left associate to $a_1 b$: $Sc = Sa_1 b$. By Lemma 2.1, $Pc = Pa_1 b$, and in particular $c \in Pb$. Similarly, from $c^{-1}c = b^{-1}b \cdot a^{-1}a$ we infer that $c \in Pa$. Thus $Pc \subseteq Pa \cap Pb$. On the other hand, let $d \in Pa \cap Pb$. Then $d \in Sa \cap Sb = Se_a \cap Se_b$. Hence $de_a = de_b = d$, and therefore $de_c = de_a e_b = d$. Thus $d \in Se_c = Sc$, and, again using Lemma 2.1, $d \in Pc$. Hence $Pa \cap Pb \subseteq Pc$, and equality follows.

Assume conversely that $Pa \cap Pb = Pc$. As noted just before the statement of the lemma, there is an element x of P such that $e_a e_b = e_b e_a = e_x$. By the foregoing, $Px = Pa \cap Pb$. Hence $Px = Pc$, and $e_x = e_c$ by Lemma 3.1.

We collect these results into the following theorem, the proof of which is evident from Lemmas 3.1 and 3.2.

THEOREM 3.1. *Let S be a semigroup satisfying* A1, 2, 3, *and let P be its right unit subsemigroup. Then P satisfies* B3 (*as well as* B1 *and* B2), *and the semi-lattice of principal left ideals of P under intersection is isomorphic with the semi-lattice of idempotent elements of S.*

4. Proof of the main theorem. Let P be a semigroup satisfying conditions B1, 2, 3. Let U be the group of units of P. If a and b are elements of P, we shall write $a \sim b$ if a and b are left associate: $Pa = Pb$.

LEMMA 4.1. *If $a \sim b$ then $a = ub$ with $u \in U$.*

Proof. From $Pa = Pb$ we have $a = xb$ and $b = ya$ with $x, y \in P$. From $a = xya$, $b = yxb$, and B1, we conclude $xy = yx = 1$. Thus, $x, y \in U$.

We note incidentally that every left or right unit of P is a (two-sided) unit. For if $ab = 1$ then $(ba)^2 = ba$ and hence $ba = 1$, since by B1 the only idempotent element of P is 1.

We now suppose that a representative is chosen in each class of left associates of P, so that the LCLM $a \vee b$ becomes definite, as in § 1. The operation $*$ is defined by (1.1).

LEMMA 4.2. $(a \vee b)c \sim ac \vee bc (a, b, c \in P)$.

Proof. $P(a \vee b)c = (Pa \cap Pb)c = Pac \cap Pbc = P(ac \vee bc)$.

LEMMA 4.3. *For any element a of P, $u_a = 1 * a$ is a unit, and $a * 1 = a \vee 1 = u_a a$.*

Proof. $a \vee 1 \sim a$, and hence $a \vee 1 = u_a a$ with $u_a \in U$ by Lemma 4.1. From $(1 * a)a = 1 \vee a = a \vee 1 = u_a a$ and B1, we conclude $1 * a = u_a$. For the rest, $a * 1 = (a * 1)1 = a \vee 1$.

LEMMA 4.4. *For $a, b \in P$ and $u, v \in U$, we have $(ua * vb)v = a * b$.*

Proof. $ua \vee vb = a \vee b$ since both are equal to the representative element of the same class of associates. Hence

$$(ua * vb)vb = ua \vee vb = a \vee b = (a * b)b$$

and the result follows from B1.

LEMMA 4.5. *If $a' = ua$, $b' = ub$, $c' = vc$, $d' = vd$ with $u, v \in U$, then $(c' * b')a' = (c * b)a$ and $(b' * c')d' = (b * c)d$.*

Proof. By Lemma 4.4,

$$(c' * b')a' = (vc * ub)ua = (c * b)a,$$

$$(b' * c')d' = (ub * vc)vd = (b * c)d.$$

We now let $P^{-1} \circ P$ be the set of ordered pairs (a, b) of elements of P, with equality defined by (1.2), and define a product in $P^{-1} \circ P$ by (1.3). By Lemma 4.5, this product is single-valued.

LEMMA 4.6.

(1) $(a, b)(1, c) = (a, bc)$, (2) $(a, 1)(1, c) = (a, c)$,

(3) $(1, b)(1, c) = (1, bc)$, (4) $(a, 1)(c, 1) = (ca, 1)$.

Proof. By (1.2), (1.3), and Lemma 4.3,

$$(a, b)(1, c) = ((1 * b)a, (b * 1)c) = (u_b a, u_b bc) = (a, bc).$$

(2) and (3) follow from (1) on setting $b = 1$ and $a = 1$, resp. To show (4):

$$(a, 1)(c, 1) = ((c * 1)a, (1 * c)1) = (u_c ca, u_c) = (ca, 1).$$

LEMMA 4.7.

$$(a', a)(1, b) \cdot (c, c') = (a', a) \cdot (1, b)(c, c').$$

Proof. Let $p = c * b$, $q = c * ab$, $r = p * a$ and $p' = b * c$, $q' = ab * c$, $r' = a * p$, so that $pb = b \vee c = p'c$, $qab = ab \vee c = q'c$, $ra = a \vee p = r'p$. Then, by Lemma 4.2,

$$rab = (a \vee p)b \sim ab \vee pb \sim ab \vee b \vee c \sim ab \vee c = qab.$$

By Lemma 4. 1, $rab = uqab$ with $u \in U$, and hence $r = uq$ by B1. Furthermore,

$$r'p'c = r'pb = rab = uqab = uq'c, \text{ whence } r'p' = uq',$$

by B1. By Lemma 4. 6 (1),

$$(a', a)(1, b) \cdot (c, c') = (a', ab)(c, c') = ((c * ab)a', (ab * c)c') = (qa', q'c').$$

On the other hand,

$$(a', a) \cdot (1, b)(c, c') = (a', a)(c * b, (b * c)c') = (a', a)(p, p'c')$$
$$= ((p * a)a', (a * p)p'c') = (ra', r'p'c') = (uqa', uq'c') = (qa', q'c').$$

LEMMA 4. 8. *The associative law holds in* $P^{-1} \circ P$.

Proof. By Lemma 4. 7, the associative law holds for any triad the middle one of which has the form $(1, b)$. The same is proved analogously if the middle term has the form $(b, 1)$. Using Lemma 4. 6 (2),

$$(a, a') \cdot (b, b')(c, c') = (a, a')[(b, 1)(1, b') \cdot (c, c')]$$
$$= \cdots = [(a, a') \cdot (b, 1)(1, b')](c, c') = (a, a')(b, b') \cdot (c, c'),$$

where the dots indicate four applications of Lemma 4. 7 or its dual.

By Lemma 4. 6 (3), the elements $(1, a)$ of $P^{-1} \circ P$ constitute a subsemigroup thereof, isomorphic with P. The next lemma shows that this is just the right unit subsemigroup of $P^{-1} \circ P$.

LEMMA 4. 9. *The elements of* $P^{-1} \circ P$ *having right inverses are just those of the form* $(1, a)$. *The right inverse of* $(1, a)$ *is* $(a, 1)$.

Proof. By Lemma 4. 1, $a \vee a = ua$ with $u \in U$. From $(a * a)a = a \vee a = ua$, we conclude $a * a = u$. Hence $(1, a)(a, 1) = (a * a, a * a) = (u, u) = (1, 1)$. Suppose conversely that $(a, b)(c, d) = (1, 1)$, that is, $((c * b)a, (b * c)d) = (1, 1)$. This implies that $(c * b)a$, and hence also a, is a unit. But then $(a, b) = (1, a^{-1}b)$.

LEMMA 4. 10. $P^{-1} \circ P$ *is* d-*simple.*

Proof. By Lemma 4. 9 and its dual, together with Lemma 4. 6 (2), every element of $P^{-1} \circ P$ is expressible as the product of a left unit by a right unit. By Theorem 2. 1, $P^{-1} \circ P$ is d-simple.

LEMMA 4. 11. *The idempotent elements of* $P^{-1} \circ P$ *are just those of the form* (a, a). *We have* $(a, a)(b, b) = (a \vee b, a \vee b)$ *and, in particular,* A3 *holds in* $P^{-1} \circ P$.

Proof. Suppose (a, b) is idempotent:

$$(a, b) = (a, b)(a, b) = ((a * b)a, (b * a)b).$$

This means that $(a * b)a = ua$, $(b * a)b = ub$ for some $u \in U$. Hence $a * b = b * a = u$. From $(a * b)b = a \vee b = (b * a)a$ we conclude that $ub = ua$, or $b = a$.

Conversely, since $a * a \in U$, we have

$$(a, a)(a, a) = ((a * a)a, (a * a)a) = (a, a).$$

If now (a, a) and (b, b) are any two idempotents of $P^{-1} \circ P$, then

$$(a, a)(b, b) = ((b * a)a, (a * b)b) = (a \vee b, a \vee b).$$

This last equation, incidentally, shows again the isomorphism between the semi-lattice of all idempotent elements of $P^{-1} \circ P$ and that of P under LCLM.

The first part of the main theorem is established by Lemmas 4. 8-4. 11. Turning to the second half, Theorem 3. 1 shows that P satisfies B1, 2, 3. We proceed to show that S is isomorphic with $P^{-1} \circ P$. By (1. 2) and Lemma 2. 3, we see that the correspondence $a^{-1}b \leftrightarrow (a, b)$ is one-to-one between S and $P^{-1} \circ P$. To prove the isomorphism, we must show that two elements $a^{-1}b$, $c^{-1}d$ of S multiply like (1. 3), i. e., $a^{-1}b \cdot c^{-1}d = [(c * b)a]^{-1}[(b * c)d]$, where $*$ is defined by (1. 1). From $c * b = (c \vee b)b^{-1}$ we have $(c * b)^{-1} = b(c \vee b)^{-1} = b(b \vee c)^{-1}$. By Lemma 3. 2, $e_{b \vee c} = e_b e_c$, and hence

$$\begin{aligned}[(c * b)a]^{-1}[(b * c)d] &= a^{-1}(c * b)^{-1}\ (b * c)d \\ &= a^{-1}b(b \vee c)^{-1} \cdot (b \vee c)c^{-1}d = a^{-1}b \cdot e_{b \vee c} \cdot c^{-1}d \\ &= a^{-1}b \cdot e_b e_c \cdot c^{-1}d = a^{-1}b \cdot c^{-1}d,\end{aligned}$$

since $be_b = b$ and $e_c c^{-1} = c^{-1}$.

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STRONGLY TOPOLOGICAL IMBEDDING OF F_σ-SUBSETS OF E_n.[1]

By HENRY SHARP, JR.

1. Introduction.[2] All spaces considered in this paper shall be separable and metric. Denote by E_n Euclidean space of n dimensions, by M_n^k the set of points in E_n having at most k rational coordinates, by L_n^k the set of points in E_n having at least k rational coordinates. Note that $L_n^k = E_n - M_n^{k-1}$, $\dim M_n^k = k$, $\dim L_n^k = n - k$ [5], p. 42. It is well known that the set M_{2k+1}^k contains a topological image of every k-dimensional point set and that the number $2k+1$ cannot be improved. It is, however, an open question whether M_n^k $(n < 2k+1)$ contains a topological image of every k-dimensional subset of E_n [5], p. 65. For the special cases $k = n$ and $k = n-1$ the answer to this question is in the affirmative; in fact, for these values of k the set M_n^k contains a *strongly topological image* (image under a homeomorphism of the space onto itself) of every k-dimensional subset of E_n.

The present paper is concerned with a related problem: does the set M_n^k contain a strongly topological image of every j-dimensional F_σ-subset of E_n? (Obviously the problem is solved for $k = n$ and $k = n-1$.) S. W. Hahn has given the answer to this problem for certain special cases, namely: affirmative for $n = 2$, $k = 0$, and $n = 3$, $k = 1$, $j = 0$; negative for $n = 3$, $k = 0$ [4], pp. 308-311. The solution of this problem is completed in the present paper in which it is shown that the answer is affirmative for $n \geqq 4$, $k = n-2$, and negative in all other previously unsolved cases. The results here together with previous results are concisely summarized in Figures 1 and 2. The presence of lines connecting a particular S_n^j with a particular M_n^k indicates that every j-dimensional F_σ-subset of E_n is strongly homeomorphic to a subset of M_n^k, while the absence of connecting lines indicates that there exists a compact j-dimensional subset of E_n which is not strongly homeomorphic to a subset of M_n^k. For example, the figure indicates that M_4^2 contains a strongly homeomorphic image of every 0-, 1-, and 2-dimensional

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[2] This paper is taken from the author's Ph. D. dissertation, Duke University, 1952.

F_σ-subset of E_4, and also that there exist both 0-, and 1-dimensional compact sets in E_4 which are not strongly homeomorphic to a subset of M_4^1.

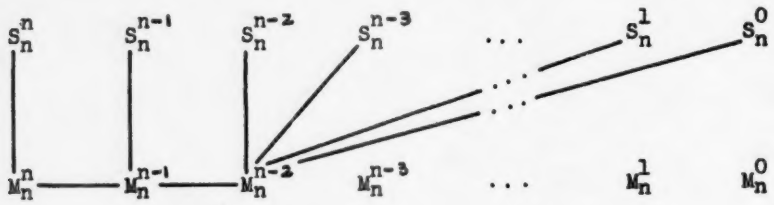

FIGURE 1 $(n = 0, n = 1, n = 2, n \geqq 4)$.

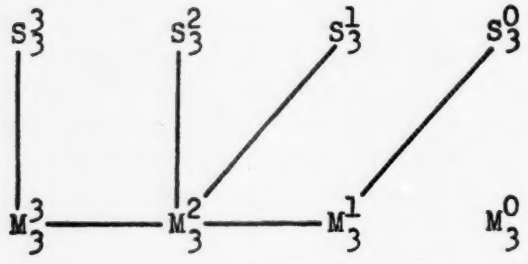

FIGURE 2.

The principal affirmative result is embodied in the theorem of Section 3, while the negative results follow from the counter-examples discussed in Sections 4 and 5. Section 2 is used to establish two lemmas which are essential in the proof of the theorem. Section 6 concludes the paper with an indication of several of the unsolved problems suggested by this investigation. The author wishes to acknowledge his indebtedness to Professor John H. Roberts for his guidance in the preparation of this paper.

2. Preliminary lemmas.

LEMMA 1. *Suppose A, B, C are distinct non-collinear points in E_n. Let $\epsilon > 0$ be given. Let $\alpha = [A, B] \cup [B, C]$, let $\lambda = [A, C]$, let σ be the 2-cell (A, B, C), let Δ be the set of points of E_n whose distance from σ is less than ϵ. Then there exists an n-polytope K_n and a homeomorphism $f(E_n) = E_n$ such that (1) $\sigma \subset K_n \subset \Delta$, (2) $f(\alpha) = \lambda$, (3) $f(P) = P$ for P belonging to $E_n - K_n$.*

Proof. We give a proof of this lemma for the case $n = 3$, and it is

evident that a continuation of the method of proof will establish the lemma for any integer $n > 3$.

In E_3 let the system of coordinates be chosen so that λ is on the x_1 axis with its midpoint, Q, at the origin of coordinates, and so that B is in the x_1x_2-plane with $B^2 > 0$ (P^i denotes the i-th coordinate of the point P). Let l be the line on the points B and Q, and let D ($D^2 > B^2$) and E ($E^2 < 0$) be points of l contained in Δ. Let K_2 denote the 2-polytope whose vertices are A, D, B, Q, E, C (as in Figure 3).

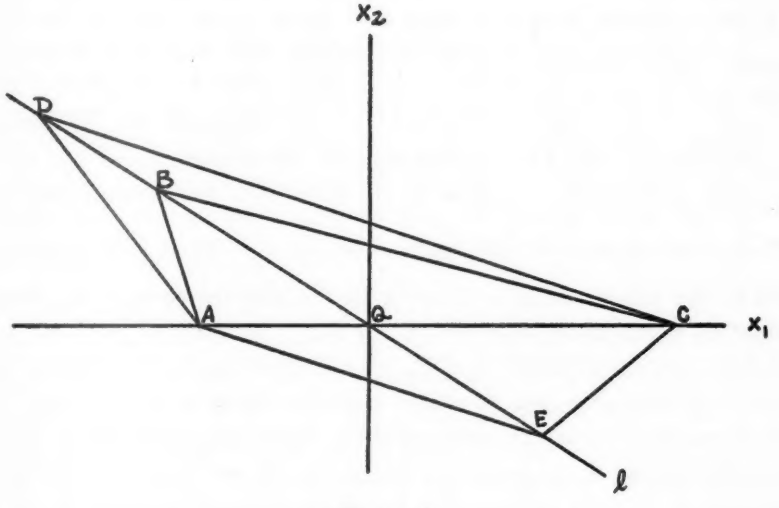

FIGURE 3.

There exists a piecewise linear transformation f_1 of the plane x_1x_2 onto itself which moves only points interior to K_2 such that $f_1[A, B] = [A, Q]$ and $f_1[B, C] = [Q, C]$, and such that $f_1(P)$ is on the line through P parallel to l.

Let F ($F^3 > 0$) and G ($G^3 < 0$) be points on the x_3-axis lying in Δ, and let K_3 be the 3-polytope which is the smallest convex set containing K_2, F, and G. There exists a piecewise linear transformation f_2 of E_3 onto itself which moves only points interior to K_3 such that $f_2(P) = f_1(P)$ if P belongs to the plane x_1x_2, and such that $f_2(P)$ is on the plane through P parallel to the x_1x_2-plane. The transformation thus described is a homeomorphism satisfying the lemma for the case $n = 3$.

If the points A, B, C are related in such a way that $A^1 < B^1 < C^1$, then by a slight modification in the above proof it is possible to add a fourth

conclusion to the lemma: (4) *if P lies on an $(n-1)$-dimensional hyperplane perpendicular to λ, then $f(P)$ lies on the same hyperplane.*

LEMMA 2. *Let X be a compact point set of dimension $\leqq n-2$ contained in E_n, $n \geqq 4$. Let L be a line in E_n and let $\epsilon > 0$ be assigned. Then there exists a homeomorphism $h(E_n) = E_n$ and a compact set D such that (1) $h(X) \cap L = 0$, (2) $d(P, h(P)) < \epsilon$ for all points P of E_n, (3) $h(P) = P$ for all points P of $E_n - D$.*

Proof. For convenience assume that L is the x_1-axis. There exists a positive number η such that $X \cap L$ is contained in the segment $(-\eta, \eta)$ on L. Let m be a positive integer so large that $2\eta/m < \epsilon/n^{\frac{1}{2}}$, and let the interval $[-\eta, \eta]$ be divided into m equal subintervals each of length δ $(= 2\eta/m)$ by the $m+1$ points $A_0, A_1, \cdots, A_m$ ordered from $A_0: (-\eta, 0, \cdots, 0)$ to $A_m: (\eta, 0, \cdots, 0)$. Let D_i, $i = 1, 2, \cdots, m$, represent the n-dimensional cube bounded by the $(n-1)$-dimensional hyperplanes $x_1 = a_{i-1}$, $x_1 = a_i$, $x_2 = \pm \frac{1}{2}\delta$, $x_3 = \pm \frac{1}{2}\delta, \cdots, x_n = \pm \frac{1}{2}\delta$, where a_i is the x_1 coordinate of the point A_i. Let $D = \bigcup_{i=1}^{m} D_i$ and let P_i, $i = 1, 2, \cdots, m-1$, be a point contained in the interior of $D - X$ lying on the hyperplane $x_1 = a_i$. For convenience of notation let $P_0 = A_0$, $P_m = A_m$. For each i, $i = 1, 2, \cdots, m$, we define a homeomorphism f_i of E_n onto itself which is the identity outside of D_i. The number m was so chosen that the diameter of D_i is less than ϵ. Since X cannot separate D_i, there exists a simple polygonal arc α_i from P_{i-1} to P_i such that (1) $X \cap \alpha_i = 0$, (2) except for end points α_i is contained in the interior of D_i, (3) the vertices of α_i are in general position; that is, no $j+2$ of the vertices lie in a j-dimensional linear subspace of E_n, $j = 0, 1, \cdots, n-1$. By using Lemma 1 a finite number of times on the vertices of α_i, each use of Lemma 1 yielding an arc having one less vertex than the previous arc, we determine a homeomorphism f_i of E_n onto itself which brings α_i into coincidence with the line interval $[P_{i-1}, P_i]$ and which moves only points interior to D_i. It should be noted here that the condition $n \geqq 4$ is essential to insure that no use of Lemma 1 shall disturb points of the arc other than those specifically under consideration. If $f = \prod_{i=1}^{m} f_i$ and $\beta = \bigcup_{i=1}^{m} [P_{i-1}, P_i]$ then (1) $f(\bigcup_{i=1}^{m} \alpha_i) = \beta$, (2) f is the identity on $E_n - D$, (3) if P is a point of D_i then $f(P)$ is a point of D_i, (4) $\beta \cap f(X) = 0$. Again there exists a homeomorphism g of E_n onto itself which is the resultant of a finite number of homeomorphisms, each leaving the x_1 coordinate invariant (applying here

conclusion 4 of Lemma 1), having the properties (1) $g(\beta) = D \cap L$, (2) g is the identity on $E_n - D$. Hence $h = gf$ is a homeomorphism of E_n onto itself such that (1) $L \cap h(X) = 0$, (2) $d(P, h(P)) < \epsilon$, (3) $h(P) = P$ for all points P of $E_n - D$.

3. The case $n \geqq 4$, $k = n - 2$.

THEOREM.[3] *The set $M_n{}^{n-2}$, $n \geqq 4$, contains an image under a homeomorphism $h(E_n) = E_n$ of every F_σ-subset of E_n of dimension $\leqq n - 2$. Furthermore, the collection $\{h\}$ of all such homeomorphisms for a given F_σ forms a dense G_δ-subset of the set of all homeomorphisms of E_n onto itself.*

The proof of this theorem rests on the fact that $L_n{}^{n-1}$ $(= E_n - M_n{}^{n-2})$ consists of a countable number of lines which can be imbedded under a strong homeomorphism in the complement of any F_σ-subset of E_n of dimension $\leqq n - 2$. Before proceding to the proof, a few preliminary remarks are necessary. Let F_n denote the set of all mappings of the n-sphere, S_n, into itself which leave a given point, ω, invariant. We metrize F_n by the distance formula $\rho(f_1, f_2) = \max_{x \in S_n} \delta(f_1(x), f_2(x))$, where f_1, f_2 are elements of F_n and δ is the usual metric on the sphere. Since F_n is a closed subset of the complete space $S_n{}^{S_n}$ it follows that F_n is complete. Let H_n denote the set of all homeomorphisms of S_n onto itself leaving ω invariant. By an argument similar to one appearing in [5], p. 57 it follows that H_n is a G_δ-subset of F_n. Hence H_n can be assigned a metric, σ, under which it becomes a complete space (see, e. g., [7], p. 29). It is well known that there is a natural one-to-one correspondence, defined by the stereographic projection, between the elements of H_n and the homeomorphisms of E_n onto itself. Hereafter we will think of H_n as being the space of all homeomorphisms of E_n onto itself as metrized by σ or by the topologically equivalent metric ρ. (It should be noted that ρ, in contradistinction to σ, is not a complete metric). We note here for future reference that if x, y are points of E_n and x', y' the corresponding points of S_n then $\delta(x', y') \leqq 2 \cdot d(x, y)$.

LEMMA. *Let X be a compact subset of E_n, $n \geqq 4$, of dimension $\leqq n - 2$, and let L be a line in E_n. Then the set G of all homeomorphisms $g(E_n) = E_n$ such that $g(X) \cap L = 0$ is both open and dense in H_n.*

Proof. If g belongs to G then since $g(X)$ is compact and L is a line

[3] This theorem, as originally stated, included only the first sentence. The author is indebted to Professor S. Eilenberg and to the referee for their indication that Baire's Theorem could be applied to yield the stronger result.

it follows by an elementary argument that G is open in H_n. Let h be an arbitrary element of H_n and let $\epsilon > 0$ be assigned. We wish to show that there exists an element g of G such that $\sigma(h, g) < \epsilon$. But since ρ and σ are topologically equivalent metrics and density is a topological property, it follows that it will be sufficient to determine g belonging to G such that $\rho(h, g) < \epsilon$. Since $h(X)$ is compact there exists, by Lemma 2 of Section 2, an element ϕ of H_n such that $\phi[h(X)] \cap L = 0$ and $d(x, \phi(x)) < \frac{1}{2}\epsilon$ for all points x, where d is the Euclidean metric. Since $\delta(x', \phi'(x)) < 2 \cdot d(x, \phi(x))$ it follows that $\rho(e, \phi) < \epsilon$, e denoting the identity element of H_n. But $\rho(h, \phi h) = \rho(e, \phi)$, hence $\rho(h, \phi h) < \epsilon$. To complete the proof we have only to set $g = \phi h$.

Proof of the theorem. Let X be an F_σ-subset of E_n of dimension $\leqq n-2$. Put $X = \bigcup_{i=1}^{\infty} X_i$, where for each i the set X_i is compact and of dimension $\leqq n-2$. The set L_n^{n-1} is the union of a countable number of lines, $L_n^{n-1} = \bigcup_{j=1}^{\infty} L_j$. Let $H^{i,j}$ denote the set of all elements h of H_n such that $h(X_i) \cap L_j = 0$. By the previous lemma $H^{i,j}$ is both open and dense in H_n. Hence, applying Baire's Theorem, the intersection of all the $H^{i,j}$ is a dense G_δ-set in H_n.

4. The case $n \geqq 4$, $k < n-2$.

LEMMA. *For $n \geqq 2$ the identity mapping f of the space L_n^{n-1} in the space L_n^{n-2} is inessential.*

Indication of proof. We wish to show that f is homotopic to zero in L_n^{n-2}. For $p: (x_1, x_2, \cdots, x_n)$ belonging to L_n^{n-1}, $0 \leqq t \leqq 1$, we define a homotopy θ_1 as follows: $\theta_1(p, t) = ((1-t)x_1, x_2, \cdots, x_n)$. Since p has at least $n-1$ rational coordinates it follows that $\theta_1(p, t)$ has at least $n-2$ rational coordinates, hence is in L_n^{n-2}. Clearly $\theta_1(p, 0) = f(p) = p$, $\theta_1(p, 1) = (0, x_2, \cdots, x_n)$ which is a point of L_n^{n-1}. Similarly, we can define a homotopy θ_2 over the set $\theta_1(L_n^{n-1}, 1)$, $0 \leqq t \leqq 1$, so that $\theta_2[\theta_1(p, 1), 1] = (0, 0, x_3, \cdots, x_n)$. A series of n such homotopies can be defined leading to the desired result.

Antoine has given an example of a compact, 0-dimensional subset of E_3 whose complement is not simply connected [1], 663. Recently, W. A. Blankinship, through a generalization of Antoine's example, has shown the existence of a compact, 0-dimensional subset of E_n, $n \geqq 3$, whose complement

is not simply connected [2]. Blankinship's result leads to the following theorem: *Not every compact, 0-dimensional subset of E_n, $n \geqq 4$, is strongly homeomorphic to a subset of M_n^{n-3}.* For let A be a compact, 0-dimensional subset of E_n such that $E_n - A$ is not simply connected. If there exists a strong homeomorphism h such that $h(A)$ is contained in M_n^{n-3}, then $E_n - h(A)$ contains L_n^{n-2}. The property of being simply connected is a topological invariant, hence there exists a closed path J in $E_n - h(A)$ which is not homotopic to zero in $E_n - h(A)$. But L_n^{n-2} consists of a countable number of planes (parallel to the coordinate planes) which are dense throughout the space. It is evident that J is homotopic in $E_n - h(A)$ to a closed path J' contained in L_n^{n-1}, and since the identity mapping of L_n^{n-1} in L_n^{n-2} is inessential, the path J must be homotopic to zero.

5. The case $n = 3$, $k = 1$. Previous results have completely solved the problem under consideration for the cases $n = 0$, $n = 1$, and $n = 2$, and have left just one question for the case $n = 3$, namely: does the set M_3^1 contain a strongly topological image of every 1-dimensional F_σ-subset of E_3? That the answer to this question is in the negative is shown by an example of a compact 1-dimensional subset of E_3 described in a note added by the referee to a paper of Frankl and Pontrjagin [3], p. 788. This example is obtained by omitting from the unit cube a countably infinite sequence of knotted canals running between opposite faces of the cube. Denote this compact, 1-dimensional set by D and suppose D can be imbedded in M_3^1 by a strong homeomorphism. Then by taking complements L_3^2 is contained in $E - f(D)$. But this is impossible since L_3^2 is the set of rational lines contained in E_3.

6. Conclusion. The only case which differs from the general pattern of results in Figure 1 is the negative result for 1-dimensional compact subsets of E_3. This divergence is intimately connected with the existence of knotted curves, which can occur only in a space of three dimensions. Thus the proof of the lemma of Section 2 breaks down if $n = 3$ since then there is no guarantee that the closed curve $\alpha_i + [P_{i-1}, P_i]$ will be free of knots.

This investigation has left unsolved several problems, of a similarly specialized nature, in connection with the general problem stated in the first paragraph of the introduction. It would be desirable to find an answer to the question: does there exist a k-dimensional subset of E_n which contains a strongly homeomorphic image of every k-dimensional compact subset of E_n, $n \geqq 3$, $k < n - 2$? Sections 4 and 5 prove that the set M_n^k does not have

this property. Another question arises on removing the restriction of strong homeomorphisms: does the set M_n^k, $n \geqq 6$, $[\frac{1}{2}n - \frac{1}{2}] < k < n - 2$ contain a topological image of every k-dimensional F_σ-subset of E_n?

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ON THE CHARACTERISTIC CLASSES OF COMPLEX SPHERE BUNDLES AND ALGEBRAIC VARIETIES.* †

By SHIING-SHEN CHERN.

Introduction. In a recent paper [1] Hodge studied the question of identifying, for non-singular algebraic varieties over the complex field, the characteristic classes of complex manifolds [2] with the canonical systems introduced by M. Eger and J. A. Todd.[3] He proved that they are identical up to a sign, when the algebraic variety is the complete intersection of non-singular hypersurfaces in a projective space. His method does not seem to extend to a general algebraic variety. One of the main difficulties lies in the fact that the theory of canonical systems of algebraic varieties has so far been developed only in broad outlines, with the result that very few of their properties are available.

We shall give in this paper a more direct treatment of the problem, by proving that there is an equivalent definition of the characteristic classes, which is valid for algebraic varieties. In order to make the paper as self-contained as possible, let us begin by recalling the original definition of the characteristic classes. We consider a compact complex manifold M_n [4] of complex dimension n, and over M_n consider the bundle B_{nr}* of ordered sets $(e_1, \cdots, e_r)$ of r linearly independent complex vectors with the same origin.[5] The fiber of this bundle is the complex Stiefel manifold V_{nr} of all the ordered sets of r linearly independent complex vectors in a complex vector space of dimension n. It is well-known that V_{nr} is connected and that its first non-vanishing homotopy group is $\pi_{2n-2r+1}(V_{nr})$, the latter being free cyclic. To describe a generator of $\pi_{2n-2r+1}(V_{nr})$ we fix $e_1, \cdots, e_{r-1}$ and let W_{n-r+1}

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[1] Hodge [9]. The number refers to the bibliography at the end of the paper.

[2] Chern [1].

[3] Eger [2] and Todd [13].

[4] In general, we shall use a subscript to denote complex dimension and a superscript to denote topological dimension. When the meaning is clear, it will be dropped to simplify notation.

[5] While we shall explain, in so far as possible, the notions which will be utilized, we shall refer to Steenrod [12] as our standard reference on fiber bundles.

be a vector space of dimension $n - r + 1$ which contains no non-trivial linear combination of $e_1, \cdots, e_{r-1}$. Let W_{n-r+1}^* be the space obtained from W_{n-r+1} by deleting its origin. Then $\pi_{2n-2r+1}(V_{nr})$ and $\pi_{2n-2r+1}(W_{n-r+1}^*)$ are naturally isomorphic (under the homomorphism induced by the inclusion mapping $W_{n-r+1}^* \subset V_{nr}$). Since W_{n-r+1}, as a complex vector space, is oriented, its orientation determines uniquely a generator of $\pi_{2n-2r+1}(W_{n-r+1}^*)$, and hence of $\pi_{2n-2r+1}(V_{nr})$. In other words, the bundle B_{nr}^* is orientable. It follows from the theory of obstructions that the primary obstruction of this bundle is a cohomology class C_{n-r+1} of (topological) dimension $2(n - r + 1)$ with integer coefficients. We call C_r, $r = 1, \cdots, n$, the *r-th characteristic class* of M. If M also denotes the fundamental homology class of the oriented manifold M, the homology class γ_r defined by the cap product $\gamma_r = C_{n-r} \cap M$ is called the *r-th characteristic homology class* of M.

All these considerations apply to the case in which M is a non-singular algebraic variety over the complex field. However, since the obstructions are defined in terms of continuous cross sections over the skeletons of a triangulation of M, it does not follow that the characteristic homology class γ_r contains as representative an algebraic cycle, that is, a cycle in the form of a finite sum $\Sigma\lambda_i V_r^i - \Sigma\mu_k V_r^k$, where V_r^i, V_r^k are algebraic sub-varieties of dimension r in M and $\lambda_i \geqq 0$, $\mu_k \geqq 0$. A main purpose of this paper is to prove that this is the case.

We proceed to enumerate our results, postponing their proofs for later sections. In the course of our discussion several theorems on the homology theory of fiber bundles and on complex sphere bundles will be used. While they are to some extent known, they are either not easily accessible or not given in a form needed for our purpose. For the sake of completeness such results will be included here.

We consider a fiber bundle $p: B \to X$, about which we make once for all the following assumptions: 1) the base space X is a finite polyhedron; 2) the fiber F is a connected finite polyhedron; 3) the fundamental group of X acts trivially on the homology groups of the fibers under consideration. The last assumption makes it possible to use these groups as coefficient groups in the homology of X. Let $r > 1$ be such that $\pi_r(F) \neq 0$, $\pi_s(F) = 0$ for all $s < r$. Then it is well-known that $\pi_r(F)$ is isomorphic to the homology group $H_r(F)$ with integer coefficients. The primary obstruction of the bundle is an element of $H^{r+1}(X, H_r(F))$. Its vanishing has an implication described by the following theorem:

THEOREM 1. *Let $\pi_r(F) \cong H_r(F)$, $r > 1$, be the first non-vanishing*

homotopy group of F. If the primary obstruction vanishes, the injection mapping $l: F \to B$ induces a homomorphism $l^*: H^r(B, H_r(F)) \to H^r(F, H_r(F))$, which is onto.

The conclusion of this theorem gives information on the "homology position" of a fiber in the bundle. Relative to a coefficient group G the simplest situation is when F is totally non-homologous to zero, i. e., when the homomorphism $l^*: H^r(B, G) \to H^r(F, G)$ is onto for all r. When G is a field, such bundles were studied by Leray and Hirsch, and the cohomology ring of the bundle is found to be isomorphic, in its additive structure, to that of the Cartesian product of the fiber and the base space. Since we are mainly interested in integer coefficients, we need the following strengthened form, due to E. H. Spanier, of the Leray-Hirsch theorem:[6]

THEOREM 2. *Let $l: F \to B$ be the injection of the fiber into the bundle. Relative to a simple coefficient system G suppose there is a homomorphism $\mu: H^r(F; G) \to H^r(B; G)$ such that $l^*\mu$ is the identity automorphism of $H^r(F; G)$ for all $r \geqq 0$. Then $H^r(B; G)$ is isomorphic with $H^r(X \times F; G)$.*

We now describe an important operation in the homology theory of fiber bundles, known as "integration over the fiber." Let $H^r(F)$, $r > 0$, be the last non-vanishing cohomology group of the fiber, so that $H^s(F) = 0$ for all $s > r$. If G is a simple system of coefficient groups (that is, a system of local groups in B on which $\pi_1(B)$ acts trivially), integration over the fiber is a homomorphism

$$\Psi: H^m(B; G) \to H^{m-r}(X; H^r(F; G)). \tag{1}$$

To define Ψ let X be triangulated and let X^k be its k-dimensional skeleton. Put $B_k = p^{-1}(X^k)$. Then it is easy to see that

$$H^m(B_s; G) = 0, \qquad 0 \leqq s \leqq m - r - 1. \tag{2}$$

From the exact sequence of the pair (B_{m-r}, B_{m-r-1}):

$$\cdots \to H^m(B_{m-r}, B_{m-r-1}; G) \xrightarrow{j^*} H^m(B_{m-r}; G) \xrightarrow{i^*} H^m(B_{m-r-1}; G) \xrightarrow{\delta^*} \cdots, \tag{3}$$

it follows that j^* is onto. To an element $u \in H^m(B; G)$ let

$$u' = i^*u \in H^m(B_{m-r}; G)$$

be the image of u under the dual homomorphism of the homomorphism induced by $i: B_{m-r} \to B$. Since j^* is onto, there exists $v \in H^m(B_{m-r}, B_{m-r-1}; G)$

[6] Hirsch [7], Leray [10], pp. 183-184, and Spanier [11].

such that $j^*v = u'$. By an isomorphism which will later be described in more detail, we see that v can be identified with an $(m-r)$-dimensional cochain of X, with coefficients in $H^r(F;G)$. It can be proved that it is a cocycle and that its cohomology class depends only on u. This class is defined to be Ψu.

This operation Ψ has a simple geometrical interpretation, when X, B, F are oriented manifolds. In this case $H^r(F;G)$ is naturally isomorphic to G, so that the coefficient group on the right-hand side of (1) can be replaced by G. Denote also by X, B, F the fundamental homology classes of these manifolds. For $u \in H^m(B;G)$, $v \in H^s(X;G)$ we define, by means of cap products, the operations

$$\begin{aligned} \mathfrak{D}_B u &= u \cap B, \\ \mathfrak{D}_X v &= v \cap X. \end{aligned} \tag{4}$$

Then we have the theorem:

THEOREM 3. *If the spaces X, B, F of a fiber bundle are oriented manifolds and $u \in H^m(B;G)$, we have*

$$p_* \mathfrak{D}_B u = \mathfrak{D}_X \Psi u. \tag{5}$$

In other words, integration over the fiber is in this case dual to the homomorphism of homology classes induced by the projection p.

The results we need next center around the theory of complex sphere or vector bundles. As the structural group of the bundle of tangent vectors of M is the general linear group $G(n)$ in n complex variables, there is an associated bundle corresponding to every subgroup of $G(n)$. We realize $G(n)$ as the group of all $n \times n$ non-singular matrices with complex elements. Let $H(n,r)$ be the subgroup of $G(n)$, consisting of all matrices (a_{ik}) for which

$$a_{ik} = 0, \quad 1 \leqq k < i \leqq r; \quad r+1 \leqq i \leqq n, \quad 1 \leqq k \leqq r, \tag{6}$$

that is, of all matrices of the form

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1,r+1} & \cdots & a_{1n} \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ 0 & & & a_{rr} & a_{r,r+1} & \cdots & a_{rn} \\ & & & & a_{r+1,r+1} & \cdots & a_{r+1,n} \\ & & 0 & & & \cdots & \\ & & & & a_{n,r+1} & \cdots & a_{nn} \end{pmatrix}.$$

Let $K(n, r)$ be the subgroup of $H(n, r)$ whose matrices satisfy the further conditions $a_{11} = \cdots = a_{rr} = 1$. Finally, let $L(n, r)$ be the subgroup of $K(n, r)$ whose matrices satisfy the additional conditions $a_{ik} = 0$, $1 \leqq i < k \leqq r$. We denote by B_{nr}, $\tilde{B}_{nr}$, and B_{nr}^* the associated bundles corresponding to the subgroups $H(n, r)$, $K(n, r)$, and $L(n, r)$ respectively. The bundle B_{nr}^* is the one introduced above, whose points are ordered sets of r linearly independent vectors $e_1(x), \cdots, e_r(x)$ with the same origin $x \in M$. Similarly, a point of $\tilde{B}_{nr}$ can be identified with a sequence of simple multivectors of the form $e_1(x), e_1(x) \wedge e_2(x), \cdots, e_1(x) \wedge \cdots \wedge e_r(x) \neq 0$, with the same origin $x \in M$. To describe the geometrical meaning of the bundle B_{nr} we need the notion of a tangent direction, which is the class of non-zero tangent vectors differing from each other by a non-zero complex factor. All the tangent directions at a point form a complex projective space of dimension $n - 1$. A point of B_{nr} can be regarded as a sequence of linear spaces of directions $L_0(x) \subset L_1(x) \subset \cdots \subset L_{r-1}(x)$ in the space of tangent directions at $x \in M$, with the subscripts indicating the dimensions of these linear spaces. We notice that these spaces are related by natural projections as follows:

$$B_{nr}^* \xrightarrow{m_{nr}} \tilde{B}_{nr} \xrightarrow{p_{nr}} B_{nr} \xrightarrow{q_{nr}} M. \tag{7}$$

Moreover, under these projections every space is a bundle over the spaces which follow it.

It is the bundle

$$q_{nr}: B_{nr} \to M, \tag{8}$$

which we are most interested. Since $H(n, r)$ is defined for $1 \leqq r \leqq n - 1$, we shall assume r to be restricted by these inequalities, thus excluding $r = n$. The fiber F_{nr} of B_{nr} is the space of all sequences of linear subspaces $L_0 \subset L_1 \subset \cdots \subset L_{r-1}$ in a complex projective space of dimension $n - 1$. Its cohomology ring with integer coefficients is generated by r two-dimensional cohomology classes and its first non-vanishing homology group of dimension > 0 is $H_2(F_{nr})$, which is free abelian with r generators. Let M' be the $(2n - 2r + 1)$-dimensional skeleton of M. The bundle B_{nr}^* has a cross-section $f: M' \to B_{nr}^*$ over M'. Then $p_{nr} \circ m_{nr} \circ f$ defines a cross-section of the bundle B_{nr} over M'. Since $2n - 2r + 1 \geqq 3$, the primary obstruction of the bundle B_{nr} is zero. It follows from Theorem 1 that the homomorphism

$$l_{nr}^*: H^2(B_{nr}; H_2(F_{nr})) \to H^2(F_{nr}; H_2(F_{nr}))$$

induced by the inclusion mapping $l_{nr}: F_{nr} \to B_{nr}$ is onto. Since $H_2(F_{nr})$ is a free abelian group with r generators, the cohomology groups $H^2(F_{nr})$, $H^2(B_{nr})$

with integer coefficients can be imbedded isomorphically into $H^2(F_{nr}; H_2(F_{nr}))$, $H^2(B_{nr}; H_2(F_{nr}))$ respectively. It follows that the induced homomorphism $l_{nr}^*: H^2(B_{nr}) \to H^2(F_{nr})$ is also onto.

This induced homomorphism has another geometrical interpretation. In fact, the fiber of the bundle $p_{nr}: \tilde{B}_{nr} \to B_{nr}$ is a Cartesian product of r complex lines, each with the origin deleted. This bundle gives rise in the base space B_{nr} to r 2-dimensional characteristic classes with integer coefficients. It can be seen that their images under l_{nr}^* generate the cohomology ring of F_{nr}. Since l_{nr}^* is multiplicative, we see that the conditions of Theorem 2 are satisfied. This leads to the conclusion that the cohomology groups of B_{nr} are isomorphic to those of $M \times F_{nr}$. In other words, the space B_{nr} has rather simple additive homology properties.

In the study of the bundles in (7) an important tool is the so-called duality theorem. To describe the situation in geometrical terms, we take, over the same base space X, two bundles B_1, B_2 of complex vector spaces of dimensions ν_1, ν_2 respectively and construct a bundle of complex vector spaces of dimension $\nu_1 + \nu_2$ over X by taking as the fiber at a point $x \in X$ the space spanned by the fibers at x of the given bundles. This bundle will be called the product of B_1 and B_2 and will be denoted by $B_1 \boxtimes B_2$. The question naturally arises as to express the characteristic classes of $B_1 \boxtimes B_2$ in terms of those of B_1 and B_2. To express this relationship we introduce, for a bundle of complex vector spaces of dimension ν, the *characteristic polynomial*

$$C(t) = \sum_{i=0}^{\nu} C_i t^i, \qquad C_0 = 1. \tag{9}$$

This is a polynomial in an auxiliary variable t, whose coefficients are the characteristic cohomology classes C_i, with the convention that the classes of dimension greater than the topological dimension of X are replaced by zero. Then we have the theorem:

THEOREM 4. (Duality theorem for complex vector bundles) *If B_1 and B_2 are two complex vector bundles over the same base space X and if $C^{(1)}(t)$ and $C^{(2)}(t)$ are their characteristic polynomials, then the characteristic polynomial of their product bundle $B_1 \boxtimes B_2$ is $C^{(1)}(t)\ C^{(2)}(t)$.*

Using the duality theorem it is easy to prove the following theorem of G. Hirsch and Wu Wen-Tsun,[7] which can be regarded as giving a new definition of the characteristic classes:

[7] Hirsch [8]; Wu, unpublished.

THEOREM 5. *Let M be a compact complex manifold of dimension n, and B_{n1}, $\tilde{B}_{n1}$ two of the associate bundles of its tangent bundle, as defined above. Let u_1 be the characteristic class of the bundle $p_{n1}: \tilde{B}_{n1} \to B_{n1}$. Then we have*

$$u_1^n = \sum_{i=1}^{n} (-1)^{i+1} q_{n1}^*(C_i) u_1^{n-i}. \tag{10}$$

This formula enables us to define the characteristic classes within the framework of homology theory. Unfortunately it does not seem to achieve our purpose of giving a definition applicable to the case when M is an algebraic variety. To obtain still another definition let us notice that the fiber F_{nr} of the bundle B_{nr} is an oriented manifold of topological dimension $r(2n-r-1)$, so that $H^{r(2n-r-1)}(F_{nr}; G)$ is naturally isomorphic to G and the homomorphism (1) can be written

$$\Psi_{nr}: H^m(B_{nr}) \to H^{m-r(2n-r-1)}(M). \tag{11}$$

Moreover, instead of the characteristic polynomial $C(t)$ we can introduce the dual characteristic polynomial

$$\bar{C}(t) = \sum_{i=1}^{n} \bar{C}_i t^i, \qquad \bar{C}_0 = 1, \tag{12}$$

defined by the condition

$$C(t)\bar{C}(t) = 1. \tag{13}$$

Obviously the polynomial $C(t)$ defines $\bar{C}(t)$, and vice versa.

In order to formulate our next theorem, we need some notations. We regard a point of $\tilde{B}_{nr}$ as a sequence of simple multivectors of the form $e_1(x), \cdots, e_1(x) \wedge \cdots \wedge e_r(x) \neq 0$, with the same origin $x \in M$. Then the sequences of multivectors having the same projection in B_{nr} are exactly the ones obtained from the last sequence by multiplying its multivectors by the non-zero complex numbers $\alpha_1, \cdots, \alpha_r$ respectively. We can therefore regard $\alpha_1, \cdots, \alpha_r$ as the coordinates in the fiber of the bundle $p_{nr}: \tilde{B}_{nr} \to B_{nr}$. The fiber is thus a Cartesian product of r complex centered affine lines, each with the origin deleted. We denote by $v_1, \cdots, v_r$ their characteristic classes in B_{nr}. From them we introduce the cohomology classes $u_1, \cdots, u_r$ according to the equations

$$v_i = u_1 + \cdots + u_i, \qquad i = 1, \cdots, r. \tag{14}$$

Then we have the theorem:

THEOREM 6. *For $1 \leqq r \leqq n-1$ the following formula holds:*

$$\Psi_{nr}(u_1^{n-2} \cdots u_{r-1}^{n-r} u_r^{2n-r}) = (-1)^n \bar{C}_{n-r+1}. \tag{15}$$

In terms of homology this formula gives, on account of Theorem 3,

$$(16)\qquad \begin{aligned}(-1)^n \mathfrak{D}_M \bar{C}_{n-r+1} &= (p_{nr})_* \mathfrak{D}_B (u_1^{n-2} \cdot \cdots \cdot u_{r-1}^{n-r} u_r^{2n-r}) \\ &= (p_{nr})_* \{(\mathfrak{D}_B u_1)^{n-2} \cdot \cdots \cdot (\mathfrak{D}_B u_{r-1})^{n-r} (\mathfrak{D}_B u_r)^{2n-r}\},\end{aligned}$$

where we write B for B_{nr}. Multiplication in the last expression means intersection of the homology classes.

Now let M be a non-singular algebraic variety in a complex projective space of higher dimension. It is well-known that B_{nr} is a non-singular algebraic variety and that the projection p_{nr} is a rational mapping. As first shown by Weil,[8] each of the classes $\mathfrak{D} u_i$, $i = 1, \cdots, r$, in B_{nr} contains a divisor class which consists of all divisors linearly equivalent to each other. Since the intersection of divisor classes always contains an algebraic cycle and since, under a rational mapping, an algebraic cycle goes into an algebraic cycle, it follows from (16) that $\mathfrak{D}_M \bar{C}_{n-r+1}$ contains an algebraic cycle, for $r = 1, \cdots, n-1$. On the other hand, $\mathfrak{D}_M \bar{C}_1$ is a homology class which contains a divisor class of divisors. We can therefore state the theorem:

THEOREM 7. *Every characteristic homology class on a non-singular algebraic variety contains an algebraic cycle.*

1. On the homology theory of fiber bundles. The homology theory of fiber bundles has been the object of study of many authors. The problem is by nature not a simple one; published accounts of it are either sketchy or need much machinery in algebraic topology. We shall give below a procedure developed by E. H. Spanier and the author[9] which has the advantage of being quite elementary and which will lead to proofs of our Theorems 1, 2, and 3. We begin by discussing some elementary facts on the homology theory of topological spaces.

Let X be a topological space, and A, B, C, D four closed subsets, such that

$$D \subset C \subset A, \qquad D \subset B \subset A.$$

Then the inclusion mapping

$$j\colon (C, D) \subset (A, B)$$

induces a homomorphism of the relative cohomology groups:

$$(17)\qquad j^*\colon H^r(A, B) \to H^r(C, D).$$

[8] Weil [16].
[9] Spanier [11].

In these cohomology groups we drop the coefficient group to simplify our notation, whenever there is no danger of confusion. The case that the subset D or both D and B are empty is not excluded. When several inclusion mappings are under consideration, we shall denote them also by i, k, or l, or distinguish them by subscripts. These induced homomorphisms have some simple properties, which have been taken as axioms by Eilenberg and Steenrod in their axiomatic treatment of homology theory.[10] The following axioms will be frequently used in our discussions:

1. The excision axiom.[11] If A, B are closed subsets of X, the homomorphism

$$j^*\colon H^r(A \cup B, B) \to H^r(A, A \cap B) \tag{18}$$

is an onto isomorphism.

2. The exactness axiom. Let

$$\delta^*\colon H^{r-1}(A) \to H^r(X, A) \tag{19}$$

be the coboundary homomorphism. Then the sequence

$$\cdots \to H^{r-1}(A) \xrightarrow{\delta^*} H^r(X, A) \xrightarrow{j^*} H^r(X) \xrightarrow{i^*} H^r(A) \xrightarrow{\delta^*} H^{r+1}(X, A) \to \cdots \tag{20}$$

is exact.

Now let A, B be closed subsets of X, such that $B \subset A \subset X$. Consider the sequence

$$H^r(A, B) \xrightarrow{j^*} H^r(A) \xrightarrow{\delta^*} H^{r+1}(X, A),$$

and define the homomorphism $\Delta^* = \delta^* j^*$. It follows from (20) that the sequence

$$\cdots \to H^r(X, A) \xrightarrow{j^*} H^r(X, B) \xrightarrow{k^*} H^r(A, B) \xrightarrow{\Delta^*} H^{r+1}(X, A) \to \cdots \tag{21}$$

is exact. It is called the *exact sequence of a triple* $B \subset A \subset X$.

---

[10] Eilenberg and Steenrod [5].

[11] This is a strengthened form of the excision axiom and is not true for general homology theory. It is true for homology theories invariant under what Eilenberg and Steenrod called relative homeomorphisms. An example is given by the Cech homology or cohomology theory for the category of compact pairs (Cf. [5], 266, Theorem 5.4). In our applications all the spaces under consideration are finite polyhedra, for which this excision axiom is valid.

LEMMA 1.1. *Let X be a topological space, and A, B, C closed subsets, such that*

$$X = A \cup B, \qquad A \cap B \subset C.$$

Then we have the isomorphism

$$H^r(X, C) \cong H^r(A, A \cap C) \oplus H^r(B, B \cap C). \tag{22}$$

To prove this, we consider the following groups related by homomorphisms, all induced by inclusion mappings:

$$\begin{array}{ccccc} H^r(X, B \cup C) & \xrightarrow{j^*} & H^r(X, C) & \xrightarrow{k^*} & H^r(B \cup C, C) \\ & & \uparrow{\scriptstyle i^*} & \nearrow{\scriptstyle l^*} & \\ & & H^r(X, A \cup C) & & \end{array}$$

The groups of the first row are taken from the exact sequence of the triple $C \subset B \cup C \subset X$, and therefore form an exact sequence. By the excision axiom, l^* is an onto isomorphism. Writing $i^* l^{*-1} = \lambda$, we have $k^* \lambda =$ identity. For $x \in H^r(X, C)$, we find $k^*(x - \lambda k^* x) = 0$, which allows us to put $x - \lambda k^* x = j^* y$, $y \in H^r(X, B \cup C)$. On the other hand, if $x = \lambda z = j^* y$, $z \in H^r(B \cup C, C)$, then $z = k^* \lambda z = k^* j^* y = 0$. It follows that $H^r(X, C)$ is a direct sum of $j^* H^r(X, B \cup C)$ and $\lambda H^r(B \cup C, C)$. λ is clearly an isomorphism (into), so that i^* is an isomorphism. By symmetry between A and B it follows that j^* is an isomorphism. Since, by the excision axiom, the homomorphisms

$$H^r(X, A \cup C) \to H^r(B, B \cap C),$$
$$H^r(X, B \cup C) \to H^r(A, A \cap C),$$

induced by the inclusion mappings are onto isomorphisms, the lemma follows.

By induction this lemma can be put in the following generalized form:

LEMMA 1.1′. *Let X be a topological space, and $A_1, \cdots, A_s$, C closed subsets, such that*

$$X = A_1 \cup \cdots \cup A_s, \qquad A_i \cap A_k \subset C, \qquad i \neq k;\ i, k = 1, \cdots, s.$$

Then

$$H^r(X, C) \cong \sum_{i=1}^{s} H^r(A_i, A_i \cap C). \tag{22′}$$

the right-hand side being a direct sum of groups.

We now consider a fiber bundle $p: B \to X$, whose base space X is a

finite connected complex of dimension n and whose fiber F is connected. Denote by X^k the k-dimensional skeleton of X, and put $B_k = p^{-1}(X^k)$. It will turn out that the relative cohomology groups $H^r(B_q, B_{q-1}; G)$ can be interpreted in a simple manner. For simplicity we shall suppose our coefficient system to be simple, an assumption which is fulfilled in all our later applications.

LEMMA 1.2. *The group $H^r(B_q, B_{q-1}; G)$ is isomorphic to the direct sum*

$$\sum H^r(p^{-1}(\bar{\sigma}), p^{-1}(\dot{\sigma}); G) \cong C^q(X; H^{r-q}(F; G)), \tag{23}$$

where the summation is over all the q-dimensional cells of X and where the group $C^q(X; H^{r-q}(F; G))$ is the group of all q-dimensional cochains of X with the coefficient group $H^{r-q}(F; G)$. If λ denotes this isomorphism of $H^r(B_q, B_{q-1}; G)$ onto $C^q(X; H^{r-q}(F; G))$, then commutativity holds in the diagram

$$\begin{array}{ccccc} H^r(B_q, B_{q-1}; G) & \xrightarrow{j^*} & H^r(B_q; G) & \xrightarrow{\delta^*} & H^{r+1}(B_{q+1}, B_q; G) \\ \lambda\downarrow & & & & \downarrow\lambda \\ C^q(X; H^{r-q}(F; G)) & & \xrightarrow{\quad\delta\quad} & & C^{q+1}(X; H^{r-q}(F; G)), \end{array} \tag{24}$$

that is, $\delta\lambda = \lambda\delta^* j^*$, where δ is the coboundary operator for the group of cochains of X.

We denote by $\bar{\sigma}_a^q$, $a = 1, \cdots, s$, the closed q-cells of X, by $\dot{\sigma}_a^q$ the set-theoretical boundary of $\bar{\sigma}_a^q$, and write $\sigma_a^q = \bar{\sigma}_a^q - \dot{\sigma}_a^q$. The latter will also denote the corresponding cell, chain, or cochain, when it is oriented. Putting $A_a = p^{-1}(\bar{\sigma}_a^q)$, $C = B_{q-1}$, we get, by applying Lemma 1.1′ to the space B_q, the direct sum decomposition

$$H^r(B_q, B_{q-1}; G) \cong \sum_{a=1}^{s} H^r(p^{-1}(\bar{\sigma}_a^q), p^{-1}(\sigma_a^q); G). \tag{23′}$$

To describe this isomorphism more explicitly, we put, for every a, $B_{q,a} = p^{-1}(X^q - \sigma_a^q)$, and consider the following homomorphisms, all induced by inclusion mappings:

$$\begin{array}{ccc} & H^r(B_q, B_{q,a}; G) & \\ {\scriptstyle i_a^*}\swarrow & & \searrow{\scriptstyle j_a^*} \\ H^r(p^{-1}(\bar{\sigma}_a^q), p^{-1}(\dot{\sigma}_a^q); G) & \xleftarrow{k_a^*} & H^r(B_q, B_{q-1}; G). \end{array}$$

By the excision axiom, i_a^* is an onto isomorphism, while, by Lemma 1.1,

j_a^* is an isomorphism. Moreover, we have $i_a^* = k_a^* j_a^*$ or $k_a^*(j_a^* i_a^{*-1})$ = identity. In the decomposition (23′), $H^r(B_q, B_{q-1}; G)$ is a direct sum of the subgroups $j_a^* i_a^{*-1} H^r(p^{-1}(\bar{\sigma}_a^q), p^{-1}(\dot{\sigma}_a^q); G)$.

Since $p^{-1}(\bar{\sigma}_a^q)$ is homeomorphic to $\bar{\sigma}_a^q \times F$, we have

$$H^r(p^{-1}(\bar{\sigma}_a^q), p^{-1}(\dot{\sigma}_a^q); G) \cong H^{r-q}(F; G) \otimes H^q(\bar{\sigma}_a^q, \dot{\sigma}_a^q; Z),$$

where Z is the additive group of integers. The latter group is isomorphic to the group of cochains $\alpha\sigma_a^q$, $\alpha \in H^{r-q}(F; G)$. If μ_a denotes this isomorphism, λ is defined componentwise by $\lambda_a = \mu_a k_a^*$.

To prove the commutativity of the diagram (24), it is sufficient to take $x \in H^r(B_q, B_{q,a}; G)$ and to prove that $\delta\lambda j_a^*(x)$ and $\lambda\delta^* j^* j_a^*(x)$, both $(q+1)$-dimensional cochains of X, have the same value for any $(q+1)$-cell σ_b^{q+1}. For this purpose it is essential to consider the following diagram:

$$\begin{array}{ccccc}
 & & H^r(B_q, B_{q,a}; G) & \xrightarrow{\Delta^*} & H^{r+1}(B_{q+1}, B_q; G) \\
 & \swarrow^{i_a^*} & \downarrow{\scriptstyle l_{ab}^*} & & \searrow^{k_b^*} \\
H^r(p^{-1}(\bar{\sigma}_a^q), p^{-1}(\dot{\sigma}_a^q); G) & & H^r(p^{-1}(\dot{\sigma}_b^{q+1}), p^{-1}(\dot{\sigma}_b^{q+1} - \sigma_a^q); G) & \xrightarrow{\Delta_1^*} & H^{r+1}(p^{-1}(\bar{\sigma}_b^{q+1}), p^{-1}(\dot{\sigma}_b^{q+1}) \\
 & \searrow_{\mu_a} & & & \swarrow_{\mu_b} \\
 & & C^q(\bar{\sigma}_a^q; H^{r-q}(F; G)) & \xrightarrow{\delta} & C^{q+1}(\bar{\sigma}_b^{q+1}; H^{r-q}(F; G)).
\end{array}$$

In this diagram i_a^*, k_b^*, and l_{ab}^* are induced by inclusion mappings, δ is the coboundary operator of the group of cochains, while $\Delta^* = \delta^* j^* j_a^*$ and Δ_1^* are the coboundary operators of the triples $B_{q,a} \subset B_q \subset B_{q+1}$ and $p^{-1}(\dot{\sigma}_b^{q+1} - \sigma_a^q) \subset p^{-1}(\dot{\sigma}_b^{q+1}) \subset p^{-1}(\bar{\sigma}_b^{q+1})$. In the last notation we adopt the convention that $\dot{\sigma}_b^{q+1} - \sigma_a^q = \dot{\sigma}_b^{q+1}$, if σ_a^q is not a face of σ_b^{q+1}. Since the second triple can be mapped into the first one by the inclusion mapping, we have $k_b^* \Delta^* = \Delta_1^* l_{ab}^*$. Since we are only interested in the values of the cochains for σ_b^{q+1}, we can restrict ourselves to the bundle over $\bar{\sigma}_b^{q+1}$. Then we have

$$\lambda\Delta^* = \mu_b k_b^* \Delta^* = \mu_b \Delta_1^* l_{ab}^*,$$

$$\delta\lambda j_a^* = \delta\mu_a k_a^* j_a^* = \delta\mu_a i_a^*.$$

It suffices to prove that the homomorphisms in the right-hand members of these two equations are identical.

If σ_a^q is not a face of σ_b^{q+1}, both homomorphisms will give zero, because $H^r(p^{-1}(\dot{\sigma}_b^{q+1}), p^{-1}(\dot{\sigma}_b^{q+1} - \sigma_a^q); G) = 0$ and δ obviously gives zero. Suppose now σ_a^q be a face of σ_b^{q+1}. By the excision axiom, the homomorphism

$$i_{ab}^*: H^r(p^{-1}(\dot{\sigma}_b^{q+1}), p^{-1}(\dot{\sigma}_b^{q+1} - \sigma_a^q); G) \to H^r(p^{-1}(\bar{\sigma}_a^q), p^{-1}(\dot{\sigma}_a^q); G)$$

induced by the inclusion mapping is an onto isomorphism. Moreover, $i_a^* = i_{ab}^* l_{ab}^*$. It suffices therefore to prove that $\mu_b \Delta_1^* = \delta \mu_a i_{ab}^*$. But then all the groups in question refer to the cell $\bar{\sigma}_b^{q+1}$ or to the bundle over it, which is homeomorphic to the Cartesian product $\bar{\sigma}_b^{q+1} \times F$. The verification of the relation in this case is trivial. This completes the proof of Lemma 1. 2.

Lemma 1. 2 can be briefly described by saying that the relative cohomology group $H^r(B_q, B_{q-1}; G)$ is isomorphic to the group of q-dimensional cochains of X with the coefficient group $H^{r-q}(F; G)$ and that the homomorphism $\delta^* j^*$ becomes then the coboundary operator under this identification. Because of our assumption that the coefficient system is simple, consideration of local coefficients is not necessary.

However, we are interested not in the relative cohomology groups $H^r(B_q, B_{q-1}; G)$, but in the absolute cohomology groups $H^r(B; G) = H^r(B_n; G)$. To derive information about them, we consider successively the groups $H^r(B_q; G)$, $q = 0, 1, \cdots, n$. All these groups are connected by homomorphisms as in the diagram:

$$
\begin{array}{ccccccc}
 & & H^r(B_n, B_{n-1}) & \xrightarrow{j^*} & H^r(B_n = B) & & \\
 & & & & \downarrow i^* & & \\
 & & & & \vdots & & \\
 & & & & \downarrow i^* & & \\
\cdots \to & H^r(B_q, B_{q-1}) & \xrightarrow{j^*} & H^r(B_q) & \xrightarrow{\delta^*} & H^{r+1}(B_{q+1}, B_q) & \to \cdots \\
 & & & & \downarrow i^* & & \\
 & & & & \vdots & & \\
 & & & & \downarrow i^* & & \\
\cdots \to & H^r(B_r, B_{r-1}) & \xrightarrow{j^*} & H^r(B_r) & \xrightarrow{\delta^*} & H^{r+1}(B_{r+1}, B_r) & \to \cdots \\
 & & & & \downarrow i^* & & \\
 & & & & \vdots & & \\
 & & & & \downarrow i^* & & \\
 & & H^r(B_0, B_{-1}) & \xrightarrow{j^*} & H^r(B_0) & \xrightarrow{\delta^*} & H^{r+1}(B_1, B_0). \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}
\tag{25}
$$

In all these cohomology groups the coefficient group is G, which is omitted for simplicity of notation. Every sequence of groups from the diagram connected by homomorphisms in the cyclic order j^*, i^*, δ^* is exact. We shall denote by $H_0^r(B_q)$ that subgroup of $H^r(B_q)$, which is the kernel of δ^*.

First a remark about the homology position of the fiber in the bundle B. We take a vertex $v \in X^0$, and identify F with $p^{-1}(v)$. This gives rise to the

inclusion mappings $l_0: F \to B_0$ and $l: F \to B$, and the induced homomorphisms

$$\begin{array}{ccc} & & H^r(B) \\ & \swarrow^{l^*} & \downarrow{\scriptstyle i^*} \\ H^r(F) & \xleftarrow{l_0^*} & H_0{}^r(B_0). \end{array} \tag{26}$$

The homomorphism l_0^* in (26) is an onto isomorphism, for $H_0{}^r(B_0)$ is clearly isomorphic to the group of 0-dimensional cohomology classes of X, with the coefficient group $H^r(F)$, which is isomorphic to $H^r(F)$.

The diagram (25) leads to a homological definition of the primary obstruction, as given by the lemma:

LEMMA 1.3. *Let $r > 1$ be the integer such that $\pi_r(F) \neq 0$ and $\pi_s(F) = 0$ for all $s < r$. Let $H_r(F; Z)$ be the coefficient group G and $\omega \in H^r(F; G)$ be the cohomology class which assigns to any $z \in H_r(F; Z)$ the element z itself. There exists $\bar{\omega} \in H^r(B_r; G)$ such that $(i^*)^r\bar{\omega} = l_0{}^{*-1}\omega$; the element $\lambda\delta^*\bar{\omega}$ is a cocycle and its cohomology class is the negative of the primary obstruction.*

Consider first an element $\phi \in H_0{}^r(B_0)$. From the relations

$$H^{r+1}(B_s, B_{s-1}) = 0, \qquad 1 < s < r+1,$$

it follows that there exists $\bar{\phi} \in H^r(B_r)$ such that $(i^*)^r\bar{\phi} = \phi$. By Lemma 1.2 we have

$$\delta(\lambda\delta^*\bar{\phi}) = \lambda\delta^* j^*\delta^*\bar{\phi} = 0,$$

which means that $\lambda\delta^*\bar{\phi}$ is a cocycle. On the other hand, $\bar{\phi}$ is determined up to an additive term j^*y, $y \in H^r(B_r, B_{r-1})$, so that $\lambda\delta^*\bar{\phi}$ is determined up to an additive term $\lambda\delta^* j^* y = \delta\lambda y$, that is, up to a coboundary. Thus the cohomology class of $\lambda\delta^*\bar{\phi}$ is completely determined.

We also remark that under our assumptions the natural homomorphism

$$f: H^r(F; G) \to \mathrm{Hom}\,(G, G), \qquad G = H_r(F; Z), \tag{27}$$

is an onto isomorphism.[12] The class ω in the statement of the Lemma is therefore well defined. From the above it follows that the same is true of the cohomology class of $\lambda\delta^*\bar{\omega}$. The verification that this is equal to the

[12] Eilenberg-MacLane [4], 808, Theorem 32.1. The theorem quoted was formulated for a star-finite complex and for homology groups with infinite cycles. A similar statement is therefore true for a closure-finite complex and for cohomology groups with infinite cocycles. Our result follows from this theorem and the further fact that $H^{r-1}(F; G) = 0$.

negative of the primary obstruction can be carried out by studying a cross-section over X^r. It is straightforward and we shall omit the details here.

We are now in a position to give a proof of Theorem 1 (Cf. Introduction). Since the primary obstruction is zero, the element $\bar{\omega} \in H^r(B_r)$ can be so chosen that $\delta^*\bar{\omega} = 0$. Since

$$H^{r+1}(B_s, B_{s-1}) = 0, \qquad s > r+1,$$

there is $\bar{\bar{\omega}} \in H^r(B)$ such that $(i^*)^{n-r}\bar{\bar{\omega}} = \bar{\omega}$, which implies

$$(i^*)^n\bar{\bar{\omega}} = (i^*)^r\bar{\omega} = l_0^{*-1}\omega.$$

By the definition of our notation for inclusion mappings, we can write i^* for $(i^*)^n$, and the last relation becomes $l_0^* i^*\bar{\bar{\omega}} = l^*\bar{\bar{\omega}} = \omega$. Thus ω belongs to the image of l^*.

To prove that the same is true of any element $\phi \in H^r(F; G)$, we consider its corresponding endomorphism $f(\phi)$ of G into itself. This endomorphism $f(\phi)$ of the coefficient group induces endomorphisms ϕ_B and ϕ_F of $H^r(B; G)$ and $H^r(F; G)$ respectively. Moreover, it is clear that $\phi_F(\omega) = \phi$. From the commutativity of the diagram

$$\begin{array}{ccc} H^r(B; G) & \xrightarrow{\phi_B} & H^r(B; G) \\ l^* \downarrow & & \downarrow l^* \\ H^r(F; G) & \xrightarrow{\phi_F} & H^r(F; G) \end{array}$$

it follows that ϕ belongs to the image of l^*. This completes the proof of Theorem 1.

The following Lemma describes the operation of "integration over the fiber":

LEMMA 1.4. *Let $H^r(F; Z) \neq 0$, $r > 0$, be the last non-vanishing cohomology group of F, so that $H^s(F; Z) = 0$ for all $s > r$. To any coefficient group G an integration over the fiber*

$$\Psi\colon H^m(B; G) \to H^{m-r}(X; H^r(F; G)), \tag{1}$$

can be defined.

This homomorphism Ψ has been described in the Introduction. To prove its existence we first notice that our assumptions imply

$$H^{m-1}(B_{m-r-2}; G) = H^m(B_{m-r-1}; G) = 0.$$

Let $u \in H^m(B)$ and $u' = (i^*)^{m-r} u \in H^m(B_{m-r})$ and consider the cohomology groups

$$H^{m-1}(B_{m-r-1}, B_{m-r-2}) \xrightarrow{j^*} H^{m-1}(B_{m-r-1})$$
$$i^* \downarrow$$
$$0$$

$$\xrightarrow{\delta^*} H^m(B_{m-r}, B_{m-r-1}) \xrightarrow{j^*} H^m(B_{m-r}) \xrightarrow{\delta^*} H^{m+1}(B_{m-r+1}, B_{m-r})$$
$$i^* \downarrow$$
$$0$$

where the sequences in the cyclic order j^*, i^*, δ^* are exact. Since $i^*u' = 0$, there is $v \in H^m(B_{m-r}, B_{m-r-1})$ with $u' = j^*v$. Then $\lambda v \in C^{m-r}(X; H^r(F; G))$ is a cocycle, for $\delta\lambda v = \lambda\delta^* j^* v = \lambda\delta^* u' = 0$. Moreover, v is defined up to an additive term $\delta^* y$, $y \in H^{m-1}(B_{m-r-1})$, where $y = j^*z$, $z \in H^{m-1}(B_{m-r-1}, B_{m-r-2})$. Since $\lambda\delta^* y = \lambda\delta^* j^* z = \delta\lambda z$, λv is defined up to a coboundary. Its cohomology class, which is an element of $H^{m-r}(X; H^r(F; G))$, is therefore completely determined by u. We call it Ψu and thus prove the existence of the homomorphism Ψ.

Concerning the homomorphism Ψ, we have the following useful lemma which follows immediately from its definition:

LEMMA 1.5. *Let $p: B \to X$ and $p': B' \to X'$ be two fiber bundles and $\tilde{f}: B \to B'$ be a bundle map which induces a mapping $f: X \to X'$ of the base spaces. If Ψ, Ψ' denote integrations over the fiber of the two bundles, then commutativity holds in the diagram:*

$$\begin{array}{ccc} H^m(B) & \xleftarrow{\tilde{f}^*} & H^m(B') \\ \Psi \downarrow & & \downarrow \Psi' \\ H^{m-r}(X; H^r(F)) & \xleftarrow{f^*} & H^{m-r}(X'; H^r(F)), \end{array}$$

that is, $f^*\Psi' = \Psi\tilde{f}^*$.

2. Proof of Theorem 2 (the Generalized Leray-Hirsch Theorem).

To carry out the proof we shall adopt the notations of the preceding section. By double induction on r and q, we proceed to prove the following statements:

a) There exists a homomorphism $\mu_{rq}: H_0^r(B_q) \to H^r(B)$, such that $(i^*)^{n-q}\mu_{rq}$ is the identity automorphism of $H_0^r(B_q)$;

b) $H_0^r(B_q)$ is isomorphic to the direct sum of $H_0^r(B_{q-1})$ and $H^q(X; H^{r-q}(F))$. Here we make the convention that a cohomology group of negative dimension is vacuous.

We remark that, for $q = n$, b) implies that $H^r(B)$ is isomorphic to the

direct sum $\sum_{s=0}^{n} H^s(X; H^{r-s}(F))$. Since the latter is isomorphic to $H^r(X \times F)$ by Künneth's Theorem, our theorem follows from b).

For $q = 0$ we put $\mu_{r0} = \mu l_0^*$. Since l_0^* establishes an isomorphism between $H_0^r(B_0)$ and $H^r(F)$, the fact that $l^*\mu = l_0^* i^* \mu$ is the identity automorphism of $H^r(F)$ implies that $l_0^{*-1}(l_0^* i^* \mu) l_0^* = i^* \mu_{r0}$ is the identity automorphism of $H_0^r(B_0)$. This proves a). Statement b) is obvious, if we define $H_0^r(B_{-1})$ to be zero.

Suppose μ_{ts} be defined for $t < r$ and $t = r$, $0 \leqq s \leqq q-1$, fulfilling the conditions a), b). Consider the diagram

$$\begin{array}{ccccccc}
\to H^{r-1}(B_{q-1}, B_{q-2}) & \xrightarrow{j^*} & H^{r-1}(B_{q-1}) & \xrightarrow{\delta^*} & H^r(B_q, B_{q-1}) & & \\
& & \downarrow i^* & & \Big\downarrow j^* & & \\
& & H^{r-1}(B_{q-2}) & & H^r(B_q) & \xrightarrow{\delta^*} & H^{r+1}(B_{q+1}, B_q) \to \cdots . \\
& & & & \downarrow i^* & & \\
& & & & H^r(B_{q-1}) & &
\end{array}$$

We put $\nu = (i^*)^{n-q} \mu_{r,q-1} : H_0^r(B_{q-1}) \to H_0^r(B_q)$, so that $i^* \nu =$ identity by induction hypothesis. A familiar argument proves that $H^r(B_q)$ is a direct sum of $\nu H_0^r(B_{q-1})$ and $j^* H^r(B_q, B_{q-1})$. Moreover, ν is clearly an isomorphism. By exactness the second summand fulfills the isomorphism

$$j^* H^r(B_q, B_{q-1}) \cong H^r(B_q, B_{q-1}) / \delta^* H^{r-1}(B_{q-1}).$$

From our induction hypothesis it follows that the group $H^{r-1}(B_{q-1})$ is a direct sum of $(i^*)^{n-q+1} \mu_{r-1,q-2} H_0^{r-1}(B_{q-2})$ and $j^* H^{r-1}(B_{q-1}, B_{q-2})$ of which the first summand goes to 0 under δ^*. Therefore we have

$$j^* H^r(B_q, B_{q-1}) \cap H_0^r(B_q) \cong K^r(B_q, B_{q-1}) / \delta^* j^* H^{r-1}(B_{q-1}, B_{q-2}),$$

where $K^r(B_q, B_{q-1}) \subset H^r(B_q, B_{q-1})$ is the kernel of $\delta^* j^*$. Using the isomorphism λ, we see that this group is isomorphic to $H^q(X; H^{r-q}(F))$. This proves b).

It remains to define μ_{rq} to satisfy a). For an element

$$x = \nu y \in \nu H_0^r(B_{q-1}), y \in H_0^r(B_{q-1}),$$

we set

$$\mu_{rq} x = \mu_{r,q-1} y = \mu_{r,q-1} i^* x.$$

Then $(i^*)^{n-q} \mu_{rq} = (i^*)^{n-q} \mu_{r,q-1} i^* = \nu i^*$ is the identity automorphism, since ν is an isomorphism of $H_0^r(B_{q-1})$ onto $\nu H_0^r(B_{q-1})$. To define μ_{rq} for the other summand, we need some preparations.

First we put $B' = X \times B$ and consider B' as a bundle over X with projection p' defined by $p'(x, b) = x$, $x \in X, b \in B$. To this bundle the homology theory established in § 1 can be applied, and we shall denote the notions and symbols pertaining to it by dashes. This bundle is, however, a trivial bundle and has very simple properties. In particular, we have, by Künneth's Theorem, the isomorphisms

$$H_0^r(B_q') = H_0^r(X^q \times B) \cong \sum_{s=0}^{q} H^s(X; H^{r-s}(B)),$$

$$H^r(B') = H^r(X \times B) \cong \sum_{s=0}^{n} H^s(X; H^{r-s}(B)).$$

These permit us to define the homomorphisms

$$\mu_{rq}' : H_0^r(B_q') \to H^r(B'),$$

such that $(i'^*)^{n-q}\mu_{rq}'$ is the identity.

Next we define the mappings

$$g_q : B_q \to B_q' = X^q \times B$$

by $g_q(b) = (p(b), b)$, $b \in B_q$, and write $g_n = g$. These mappings induce homomorphisms on the cohomology groups, for which there is commutativity in the diagram

$$\begin{array}{ccccc} H^r(X^q \times B, X^{q-1} \times B) & \xrightarrow{j'^*} & H^r(X^q \times B) & \xrightarrow{\delta'^*} & H^{r+1}(X^{q+1} \times B, X^q \times B) \\ \downarrow g_q^* & & \downarrow g_q^* & & \downarrow g_{q+1}^* \\ H^r(B_q, B_{q-1}) & \xrightarrow{j^*} & H^r(B_q) & \xrightarrow{\delta^*} & H^{r+1}(B_{q+1}, B_q). \end{array}$$

To $y' \in H^r(X^q \times B, X^{q-1} \times B)$, $y \in H^r(B_q, B_{q-1})$, we have $\lambda' y' \in C^q(X; H^{r-q}(B))$, $\lambda y \in C^q(X; H^{r-q}(F))$, so that we can write

$$\lambda' y' = \sum_i h_i' \sigma_i^q, \qquad h_i' \in H^{r-q}(B),$$

$$\lambda y = \sum_i h_i \sigma_i^q, \qquad h_i \in H^{r-q}(F),$$

where σ_i^q are the q-cells of X. It follows from the definition of g_q that

$$g_q^* y' = \lambda^{-1}\{\sum_i l^*(h_i')\sigma_i^q\}.$$

Conversely, because of the existence of the homomorphism μ, we can define the homomorphism

$$\rho_{rq} : H^r(B_q, B_{q-1}) \to H^r(X^q \times B, X^{q-1} \times B)$$

by the equation

$$\rho_{rq}(y) = \lambda'^{-1}\{\sum \mu(h_i)\sigma_i^q\}.$$

It has therefore the property that $g_q^* \rho_{rq} = \text{identity}$. Moreover, from the interpretation of $\delta^* j^*$ and $\delta'^* j'^*$ as coboundary operators under the isomorphisms λ and λ', we have

$$\rho_{r+1,q+1} \delta^* j^* = \delta'^* j'^* \rho_{rq}.$$

From this it follows that $\delta^* j^* y = 0$ if and only if $\delta'^* j'^* \rho_{rq} y = 0$.

To define μ_{rq} for the summand $j^* H^r(B_q, B_{q-1}) \cap H_0^r(B_q)$ of $H_0^r(B_q)$, we take $y \in H^r(B_q, B_{q-1})$ with $\delta^* j^* y = 0$. Then $j'^* \rho_{rq} y \in H_0^r(B'_q)$ and $\mu_{rq}' j'^* \rho_{rq} y \in H^r(B') = H^r(X \times B)$. Now consider the diagram

$$\begin{array}{ccc} H^r(X \times B) & \xrightarrow{g^*} & H^r(B), \\ \downarrow (i'^*)^{n-q} & & \downarrow (i^*)^{n-q} \\ H^r(X^q \times B) & \xrightarrow{g_q^*} & H^r(B_q) \end{array}$$

in which commutativity holds. We define

$$\mu_{rq} j^* y = g^* \mu_{rq}' j'^* \rho_{rq} y.$$

When we modify y by an additive term $\delta^* j^* z$, $z \in H^{r-1}(B_{q-1}, B_{q-2})$, the right-hand side will be modified by an additive term

$$g^* \mu_{rq}' j'^* \rho_{rq} \delta^* j^* z = g^* \mu_{rq}' j'^* \delta'^* j'^* \rho_{r-1,q-1} z = 0.$$

Hence μ_{rq} depends only on $j^* y$ and not on the choice of y. Since

$$(i^*)^{n-q} \mu_{rq} j^* y = (i^*)^{n-q} g^* \mu_{rq}' j'^* \rho_{rq} y = g_q^* (i'^*)^{n-q} \mu_{rq}' j'^* \rho_{rq} y$$

$$= g_q^* j'^* \rho_{rq} y = j^* g_q^* \rho_{rq} y = j^* y,$$

we conclude that the homomorphism μ_{rq} satisfies condition a).

This completes our induction and hence the proof of Theorem 2.

The following corollary follows immediately from the above proof.

COROLLARY 2.1. *Suppose the hypotheses of Theorem 2 be satisfied. Then $H^m(B)$ is isomorphic to the direct sum $\sum_{q=0}^{m} H^q(X; H^{m-q}(F))$. For $u \in H^m(B)$, its image Ψu under the integration over the fiber Ψ is the component of u in the summand $H^{m-r}(X; H^r(F))$, where r is defined by Lemma 1.4.*

Under some further assumptions which will be satisfied in our applications, we can express these relations in a more explicit form, using our homomorphism μ. In fact, from our proof of Theorem 2, we see that $H^r(B)$ is a direct sum of $\mu_{rq}(j^* H^r(B_q, B_{q-1}) \cap H_0^r(B_q))$ for $q = 0, 1, \cdots, r$.

Suppose that our coefficient system is a ring R with unit element. We call an element of $Z^q(X; H^{r-q}(F; R))$ a cocycle of the first kind, if it is equal to a finite sum of the form $\sum_i c_i \otimes z_i$, where $c_i \in Z^q(X; R)$, $z_i \in H^{r-q}(F; R)$. If this is not the case, we call it a cocycle of the second kind. We now make the assumption that every element of $H^q(X; H^{r-q}(F; R))$ has as representative a cocycle of the first kind. This condition is satisfied when, for instance, R is a field.

To an element $y \in H^r(B_q, B_{q-1})$, with $\delta^* j^* y = 0$, we can write

$$\lambda y = \sum_i c_i \otimes z_i,$$

where $c_i \in Z^q(X; R)$, $z_i \in H^{r-q}(F; R)$. The homomorphism μ_{rq}' can be defined such that

$$\mu_{rq}' j'^* \rho_{rq} y = \sum_i \gamma_i \otimes \mu(z_i),$$

where γ_i is the class of c_i. By definition, $\mu_{rq} j^* y$ is the image of this element under the homomorphism g^*. Now we can decompose g as a product of two mappings: $g = h\Delta$, where $\Delta: B \to B \times B$ is the diagonal map defined by $\Delta(b) = (b, b)$, $b \in B$, and $h: B \times B \to X \times B$ is defined by $h(b, b') = (p(b), b')$, $b, b' \in B$. It follows that

$$g^*(\sum_i \gamma_i \otimes \mu(z_i)) = \Delta^* h^*(\sum_i \gamma_i \otimes \mu(z_i)) = \Delta^*(\sum_i p^*(\gamma_i) \otimes \mu(z_i)) = \sum_i p^*(\gamma_i) \cup \mu(z_i)$$

These considerations lead to the following theorem:

COROLLARY 2.2. *Under the hypotheses of Theorem 2 suppose further that, for an integer $m \geqq 0$, every element of $H^q(X; H^{m-q}(F; R))$ has as representative a cocycle of the first kind. Then every element $u \in H^m(B; R)$ can be written in the form*

$$u = \sum_i \mu(z_i) \cup p^*(\gamma_i), \qquad z_i \in H^q(F; R), \gamma_i \in H^{m-q}(X; R),$$

where $\dim z_i + \dim \gamma_i = m$. *If r is the integer such that $H^r(F; R)$ is the highest non-vanishing cohomology group of F with the coefficient group R, then*

$$\Psi u = \sum_{\dim z_i = r} z_i \otimes \gamma_i.$$

3. The case of manifolds; proof of Theorem 3. Throughout this section we suppose that B, X, F are oriented manifolds, with the convention that these same symbols denote their fundamental homology and cohomology classes. It is easily seen that $\Psi(B) = \pm X$. We supose that the orientations

of the manifolds are such that the positive sign holds. Under our assumptions the highest non-vanishing cohomology group of F is $H^s(F; G)$, where s is the dimension of F, and it is naturally isomorphic to G. Integration over the fiber can therefore be considered as a homomorphism

$$\Psi\colon H^m(B; G) \to H^{m-s}(X; G). \tag{28}$$

LEMMA 3.1. *Let the abelian groups G_1 and G_2 be paired to G, and let $v \in H^q(X; G_1)$, $u \in H^r(B; G_2)$. Then*

$$\Psi(p^*v \cup u) = v \cup \Psi u. \tag{29}$$

To prove this lemma we denote by $d\colon X \to X \times X$ the diagonal map defined by $d(x) = (x, x)$, $x \in X$. Similarly, let $\Delta\colon B \to B \times B$ be the diagonal map of B. We define the mappings

$$p'\colon B \times B \to X \times B, \tag{30}$$

$$p''\colon X \times B \to X \times X, \tag{31}$$

by $p'(b, b') = (p(b), b')$, $p''(x, b) = (x, p(b))$, $b, b' \in B$, $x \in X$. These give rise to two bundles, both with fibers F. We put $\Gamma = p'\Delta$. Then $\Gamma\colon B \to X \times B$ is a bundle map of the given bundle $p\colon B \to X$ into the bundle (31). By Lemma 1.5, we have commutativity in the diagram ($m = q + r$):

$$\begin{array}{ccc} & & H^m(B \times B; G) \\ & \overset{\Delta^*}{\swarrow} & \uparrow {\scriptstyle p'^*} \\ H^m(B; G) & \xleftarrow{\Gamma^*} & H^m(X \times B; G) \\ \downarrow {\scriptstyle \Psi} & & \downarrow {\scriptstyle \Psi''} \\ H^m(X; G) & \xleftarrow{d^*} & H^m(X \times X; G) \end{array}$$

where Ψ'' is the integration over the fiber of the bundle (31).

Now let u', v' be representative cocycles of the cohomology classes u, v respectively. Then $v' \times u' \in Z^m(X \times B; G)$, the group of all m-cocycles of $X \times B$. If we denote by the same symbols p', Δ cellular approximations of the respective continuous mappings, we have

$$\Delta^* p'^*(v' \times u') = \Delta^*(p^*v' \times u') = p^*v' \cup u'.$$

This proves that the class containing $v' \times u'$ is mapped by $\Delta^* p'^*$ into $p^*v \cup u$. On the other hand, this class is mapped by Ψ'' into the class containing $v' \times \Psi u'$, which goes to $v \cup \Psi u$ under d^*. From the commutativity of our diagram follows therefore the formula (29).

We are now in a position to prove Theorem 3. As stated in the Theorem,

we have $u \in H^m(B; G)$. Then $\Psi u \in H^{m-s}(X; G)$ and $\mathfrak{D}_X \Psi u \in H_{n+s-m}(X; G)$. We take any element $v \in H^{n+s-m}(X; \text{Char } G)$, and denote by a dot the multiplication of homology and cohomology classes. Then we have

$$v \cdot \mathfrak{D}_X \Psi u = v \cdot (\Psi u \cap X) = (v \cup \Psi u) \cdot X,$$
$$v \cdot p_* \mathfrak{D}_B u = (p^* v) \cdot (\mathfrak{D}_B u) = (p^* v) \cdot (u \cap B) = (p^* v \cup u) \cdot B$$
$$= \{\Psi(p^* v \cup u)\} \cdot X.$$

By Lemma 3.1, the right-hand sides of the two equations are equal. The same is therefore true of the left-hand sides, with arbitrary v. This proves Theorem 3.

4. On the bundles associated to a complex manifold. We now turn our attention to the study of a compact complex manifold M of (complex) dimension n. Some of the more important bundles associated to M have been described in the Introduction, and we shall follow the notations. This section will be devoted to the proof of a few elementary properties of these bundles.

LEMMA 4.1. *To $x \in M$ let $F_{nr} = q_{nr}^{-1}(x)$, $\tilde{F}_{nr} = (q_{nr} p_{nr})^{-1}(x)$ be the fibers of the bundles B_{nr}, $\tilde{B}_{nr}$ respectively. Then $p_{nr}(x)\colon \tilde{F}_{nr} \to F_{rn}$ is a bundle having as fiber the Cartesian product of r complex lines, each with the origin deleted. It gives rise to r characteristic cohomology classes $w_1, \cdots, w_r$ of dimension 2 in the base space F_{nr}, which generate, by ring operations, the cohomology ring of F_{nr} with integer coefficients and which can be so chosen that $w_1^{n-1} \cup \cdots \cup w_r^{n-r}$ is the fundamental cohomology class of F_{nr}.*

Let V_n be the tangent complex vector space at x, with 0 denoting its origin. A point of $\tilde{F}_{nr}$ is then a sequence of simple multivectors of V_n of the form $e_1, e_1 \wedge e_2, \cdots, e_1 \wedge \cdots \wedge e_r \neq 0$, while a point of F_{nr} is a sequence of linear spaces of directions $L_0 \subset L_1 \subset \cdots \subset L_{r-1}$. The projection $p_{nr}(x)$ assigns to the multivector $e_1 \wedge \cdots \wedge e_i$, $i = 1, \cdots, r$, the linear subspace L_{i-1} it determines. Since two simple multivectors determine the same L_{i-1} when and only when they differ from each other by a non-zero complex factor, the fiber of the bundle $p_{nr}(x)\colon \tilde{F}_{nr} \to F_{nr}$ is homeomorphic to a Cartesian product $E_1 \times \cdots \times E_r$ of r complex lines, each with the origin deleted. Moreover, we can suppose E_i to be the space of all $e_1 \wedge \cdots \wedge e_i$ having the same projection L_{i-1}. This bundle is orientable, since the spaces involved are complex manifolds. We denote by w_i the characteristic class corresponding to the factor E_i of the fiber. The classes $w_1, \cdots, w_r$ can be considered to be with integer coefficients.

To prove the assertions on the homology structure of F_{nr} we proceed by induction on r. For $r = 1$, $\tilde{F}_{n1} = V_n - 0$ and F_{n1} is the complex projective space P_{n-1} of dimension $n - 1$. In this case we see easily that w_1 is the cohomology class dual to the homology class having as representative a hyperplane of P_{n-1}, so that the Lemma is true. Suppose that the Lemma holds for $r - 1$. Let

$$t: F_{nr} \to F_{n,r-1} \tag{32}$$

be the mapping which sends the sequence $L_0 \subset L_1 \subset \cdots \subset L_{r-1}$ to $L_0 \subset L_1 \subset \cdots \subset L_{r-2}$. This defines a fiber bundle having as fiber the set of all L_{r-1} through a fixed L_{r-2} in P_{n-1}, which is homeomorphic to the complex projective space P_{n-r} of dimension $n - r$. Denote by $k_{nr}: P_{n-r} \to F_{nr}$ the inclusion mapping of a fiber into the bundle. When, in the bundle $\tilde{F}_{nr} \to F_{nr}$, we restrict the fiber to the product $E_1 \times \cdots \times E_{r-1}$, it is induced by the mapping t. Hence the first $r - 1$ characteristic classes in F_{nr} are $w_i = t^* w_i'$, $i = 1, \cdots, r - 1$, where w_i' are the characteristic classes in $F_{n,r-1}$ of the bundle $\tilde{F}_{n,r-1} \to F_{n,r-1}$. Let w_r be the class corresponding to the fiber E_r. Then $k_{nr}{}^* w_r = w$ is a generating class of P_{n-r}. We define $\mu(w^j) = w_r{}^j$, $j = 1, \cdots, n - r - 1$, and extend μ by linearity into a homomorphism of the cohomology groups of P_{n-r} into those of F_{nr} (with integer coefficients). Then $k_{nr}{}^* \mu =$ identity, and the hypotheses of Theorem 2 are satisfied. If Ψ is the integration over the fiber of the bundle (32), we see easily that

$$\Psi(w_1{}^{n-1} \cup \cdots \cup w_r{}^{n-r}) = (w_1')^{n-1} \cup \cdots \cup (w_{r-1}')^{n-r+1}.$$

Since the right-hand side of this equation is the fundamental cohomology class of $F_{n,r-1}$ by our induction hypothesis, it follows that $w_1{}^{n-1} \cup \cdots \cup w_r{}^{n-r}$ is the fundamental cohomology class of F_{nr}. This completes the proof of the Lemma.

LEMMA 4.2. *The bundle B_{nr} satisfies the hypotheses of Theorem 2.*

As in the proof of Lemma 4.1, the bundle $\tilde{B}_{nr} \to B_{nr}$ has as fiber the Cartesian product of r complex lines, each with the origina deleted. Let $v_1, \cdots, v_r$ be the corresponding characteristic cohomology classes. Since the bundle $p_{nr}(x): \tilde{F}_{nr} \to F_{nr}$ is induced by the inclusion mapping $l_{nr}: F_{nr} \to B_{nr}$, we have $l_{nr}{}^* v_i = w_i$, $i = 1, \cdots, r$. Defining $\mu w_i{}^{\rho_i} = v_i{}^{\rho_i}$, $\rho_i = 1, \cdots, n - i$, and extending it by linearity into a homomorphism of the cohomology groups of F_{nr} into those of B_{nr}, we have $l_{nr}{}^* \mu =$ identity. Hence the hypotheses of Theorem 2 are fulfilled.

5. Proof of the duality theorem. The proof of the duality theorem depends on the consideration of the universal bundles. Let $E_{\nu+N}$ be a complex vector space of dimension $\nu+N$ and let $H(\nu, N)$ be the Grassmann manifold of all ν-dimensional linear spaces through the origin in $E_{\nu+N}$. Over $H(\nu, N)$ as base space we can define a bundle of complex vectors by taking the space $S(\nu, N)$ of all pairs (x, E_ν), where E_ν is a ν-dimensional linear subspace of $E_{\nu+N}$ and $x \in E_\nu$, and defining the projection $P_{\nu N}: S(\nu, N) \to H(\nu, N)$ by $P_{\nu N}(x, E_\nu) = E_\nu$. For applications to algebraic geometry it is advantageous to identify all non-zero vectors of $E_{\nu+N}$ which differ from each other by a non-zero complex factor and hence to consider the complex projective space $P_{\nu+N-1}$ of dimension $\nu+N-1$ and the Grassmann variety $G(\nu-1, N)$ of all linear spaces of dimension $\nu-1$ in $P_{\nu+N-1}$. Since $H(\nu, N)$ and $G(\nu-1, N)$ are homeomorphic in a natural way, this does not make any difference for our present purpose, and we shall use $H(\nu, N)$ as the base space of the universal bundle.

We also remark that the characteristic classes described in the Introduction can be defined for any bundle of complex vector spaces having as base space a finite polyhedron. The bundle described in the last section is called universal, because of the following theorem:[13]

Any bundle of complex vector spaces of dimension ν over a finite polyhedron X is induced by a mapping $f: X \to H(\nu, N)$, if $\dim X \leqq 2N$. The bundles induced by two such mappings are equivalent if and only if the mappings are homotopic. If Γ_i, $i = 1, \cdots, \nu$, are the characteristic classes of the universal bundle, then those of the induced bundle are $f^*\Gamma_i$.

On the other hand, the characteristic classes of the universal bundle in $H(\nu, N)$ can be described in a simple way in terms of the homology properties of $H(\nu, N)$. The latter have been studied by Ehresmann,[14] whose results can be summarized as follows: We take a sequence of integers

$$0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_\nu \leqq N,$$

and, corresponding to such a sequence, a sequence of linear spaces through the origin 0 in $E_{\nu+N}$:

$$0 \subset L_1 \subset L_2 \subset \cdots \subset L_\nu, \tag{33}$$

such that $\dim L_i = a_i + i$, $i = 1, \cdots, \nu$. A Schubert variety, to be denoted by $(a_1 \cdots a_\nu)$, consists of all the E_ν satisfying the conditions

$$\dim (L_i \cap E_\nu) \geqq i, \qquad i = 1, \cdots, \nu. \tag{34}$$

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[13] Chern [1], 88-89, Theorems 1 and 2.

[14] Ehresmann [3].

It is of dimension $\sum_{i=1}^{p} a_i$. The Schubert varieties have the following properties, which we shall use and which we state here without proof:

1) Every Schubert variety $(a_1 \cdots a_p)$ carries a cycle, whose homology class depends only on the symbol and not on the choice of the sequence (33). All these homology classes form a homology base of $H(p, N)$. From this it follows that all odd-dimensional Betti numbers of $H(p, N)$ are zero and that there are no torsion coefficients.

2) The intersection number $KI((a_1 \cdots a_p), (N - a_p, \cdots, N - a_1))$ is equal to one. All other intersection numbers of Schubert varieties of complementary dimensions are zero.

3) Let $(a_1 \cdots a_p)$ denote also the cohomology class whose value is one for $(a_1 \cdots a_p)$ and is zero for all other homology classes. Then Γ_k is the class $(0 \cdots 0\ 1 \cdots 1)$ consisting of $p - k$ zeros followed by k ones.

After these preparations we proceed to the proof of Theorem 4.[15] Let $E_{p_1+N_1}$ and $E_{p_2+N_2}$ be complex vector spaces and let $H(p_1, N_1)$, $H(p_2, N_2)$ be the corresponding Grassmann manifolds. Let $f_\alpha\colon X \to H(p_\alpha, N_\alpha)$, $\alpha = 1, 2$, be the mappings which induce the bundles B_α over X. We denote by $E_{p_1+p_2+N_1+N_2}$ the vector space spanned by $E_{p_1+N_1}$ and $E_{p_2+N_2}$, and by $H(p_1 + p_2, N_1 + N_2)$ the Grassmann manifold of all linear spaces of dimension $p_1 + p_2$ through the origin in $E_{p_1+p_2+N_1+N_2}$. An $E_{p_1} \subset E_{p_1+N_1}$ and an $E_{p_2} \subset E_{p_2+N_2}$ span an $E_{p_1+p_2} \subset E_{p_1+p_2+N_1+N_2}$. This defines a mapping

$$F\colon H(p_1, N_1) \times H(p_2, N_2) \to H(p_1 + p_2, N_1 + N_2). \tag{35}$$

Then the bundle $B_1 \boxtimes B_2$ over X is induced by the mapping $F \circ f$, where

$$f\colon X \to H(p_1, N_1) \times H(p_2, N_2) \tag{36}$$

is defined by $f(x) = (f_1(x), f_2(x))$, $x \in X$. Denote by $\Gamma_1^{(1)}, \cdots, \Gamma_{p_1}^{(1)}$; $\Gamma_1^{(2)}, \cdots, \Gamma_{p_2}^{(2)}$; $\Gamma_1, \cdots, \Gamma_{p_1+p_2}$ the characteristic classes of the universal bundles over $H(p_1, N_1)$, $H(p_2, N_2)$, $H(p_1 + p_2, N_1 + N_2)$ respectively. We assert that the duality theorem is a consequence of the formula

$$F^*\Gamma_k = \sum_{\substack{0 \le i \le k \\ k - p_2 \le i \le p_1}} \Gamma_i^{(1)} \otimes \Gamma_{k-i}^{(2)}, \qquad k = 1, \cdots, p_1 + p_2. \tag{37}$$

In fact, we have

$$C_k = f^*F^*\Gamma_k, \qquad C_i^{(1)} = f_1^*\Gamma_i^{(1)}, \qquad C_j^{(2)} = f_2^*\Gamma_j^{(2)},$$

$$C_i^{(1)} \cup C_j^{(2)} = (f_1^*\Gamma_i^{(1)}) \cup (f_2^*\Gamma_j^{(2)}) = f^*(\Gamma_i^{(1)} \otimes \Gamma_j^{(2)}).$$

[15] The idea of this proof was indicated by Wu: cf. Wu [17].

Applying the dual homomorphism f^* to (37), we get

$$C_k = \sum_{\substack{0 \leqq i \leqq k \\ k-\nu_2 \leqq i \leqq \nu_1}} C_i^{(1)} \cup C_{k-i}^{(2)}, \tag{38}$$

which is obviously an equivalent formulation of the duality theorem.

To establish (37) it suffices to prove that its two sides, being cohomology classes of $H(\nu_1, N_1) \times H(\nu_2, N_2)$, have the same value for a homology base of dimension k of $H(\nu_1, N_1) \times H(\nu_2, N_2)$. Such a base is provided by the products of Schubert varieties of $H(\nu_1, N_1)$ and $H(\nu_2, N_2)$, whose dimensions have the sum k. Consider first a product $\zeta_1 \times \zeta_2$, of which at least one factor is not of the form $(0 \cdots 0\ 1 \cdots 1)$. The value of the right-hand side of (37) for $\zeta_1 \times \zeta_2$ is then zero, while we have

$$(F^*\Gamma_k) \cdot (\zeta_1 \times \zeta_2) = \Gamma_k \cdot F(\zeta_1 \times \zeta_2) = KI(\eta_k, F(\zeta_1 \times \zeta_2)),$$

where

$$\eta_k = (\underbrace{N_1 + N_2 - 1 \cdots N_1 + N_2 - 1}_{k}\ N_1 + N_2 \cdots N_1 + N_2). \tag{39}$$

To prove that this intersection number is zero, it suffices to assume the two Schubert varieties to be in general position and show that they have no element in common. In fact, from our assumption on $\zeta_1 \times \zeta_2$, each of the elements of $F(\zeta_1 \times \zeta_2)$ passes through a fixed linear space A of dimension $\nu_1 + \nu_2 - k + 1$. On the other hand, a linear space $E_{\nu_1+\nu_2}$ belongs to η_k, if and only if it has a linear space of dimension $\geqq k$ in common with a fixed linear space B of dimension $N_1 + N_2 + k - 1$. We can choose B so that A and B have only the zero vector in common. If there is an $E_{\nu_1+\nu_2}$ through A which has a linear space of dimension $\geqq k$ in common with B, it would follow that $A \cap B$ is of dimension $\geqq 1$. It follows therefore that the intersection number in question is zero.

It remain to take $\zeta_1 = (0 \cdots 1\ \underbrace{1 \cdots 1}_{i})$, $\zeta_2 = (0 \cdots 0\ \underbrace{1 \cdots 1}_{k-i})$, for a fixed i, and to prove that, under this choice,

$$KI(\eta_k, F(\zeta_1 \times \zeta_2)) = 1.$$

Let L_{ν_1-i}, L_{ν_1+1}, M_{ν_2-k+i}, M_{ν_2+1} be linear spaces, with subscripts indicating their dimensions, which are used to define ζ_1, ζ_2, so that

$$L_{\nu_1-i} \subset L_{\nu_1+1} \subset E_{\nu_1+N_1},$$

$$M_{\nu_2-k+i} \subset M_{\nu_2+1} \subset E_{\nu_2+N_2}.$$

By the condition (34) for Schubert varieties the elements $V_{\nu_1} \varepsilon \zeta_1$, $V_{\nu_2} \varepsilon \zeta_2$ are respectively characterized by

$$L_{\nu_1-i} \subset V_{\nu_1} \subset L_{\nu_1+1},$$

$$M_{\nu_2-k+i} \subset V_{\nu_2} \subset M_{\nu_2+1}.$$

An element $V_{\nu_1+\nu_2} \varepsilon F(\zeta_1 \times \zeta_2)$ is spanned by V_{ν_1} and V_{ν_2}. The condition that it belongs also to η_k is

$$\dim (V_{\nu_1+\nu_2} \cap B) \geqq k.$$

Since the L's, M's, and B are supposed to be in general position, B has only the zero vector in common with the space $C_{\nu_1+\nu_2-k}$ of dimension $\nu_1 + \nu_2 - k$ spanned by L_{ν_1-i} and M_{ν_2-k+i}. There exists therefore a space $D_{N_1+N_2+k}$ of dimension $N_1 + N_2 + k$, which contains B and is supplementary to $C_{\nu_1+\nu_2-k}$ in $E_{\nu_1+\nu_2+N_1+N_2}$. The projections of L_{ν_1+1} and M_{ν_2+1} in $D_{N_1+N_2+k}$ from $C_{\nu_1+\nu_2-k}$ are of dimensions $i+1$ and $k-i+1$ respectively, which we denote by L_{i+1}' and M_{k-i+1}'. Their intersections with B, say L_i'' and M_{k-i}'', are then of dimensions i and $k-i$ respectively. For a $V_{\nu_1+\nu_2}$ spanned by $V_{\nu_1} \varepsilon \zeta_1$, $V_{\nu_2} \varepsilon \zeta_2$ to belong to η_k, or, what is the same, to have an intersection of dimension $\geqq k$ with B, it is therefore necessary and sufficient that it contains L_i'' and M_{k-i}''. Such a $V_{\nu_1+\nu_2}$ is uniquely determined, as the space spanned by L_{ν_1-i}, M_{ν_2-k+i}, L_i'', M_{k-i}''. This proves that $F(\zeta_1 \times \zeta_2)$ and η_k have in common exactly one $V_{\nu_1+\nu_2}$.

Using a coordinate system, we can study, in the differentiable manifold $H(\nu_1 + \nu_2, N_1 + N_2)$, the submanifolds η_k and $F(\zeta_1 \times \zeta_2)$ in the neighborhood of their intersection. It is easily seen that the tangent spaces of these submanifolds are skew to each other, so that the points of intersection is to be counted simply. This completes the proof of formula (37) and hence the duality theorem.

6. Proofs of Theorems 5 and 6. Theorem 5 is an easy consequence of the duality theorem. To prove it we put an Hermitian metric on M, so that the structural group of the tangent bundle reduces to the unitary group. To a point $y \varepsilon B_{n1}$, that is, a tangent direction of M, let $T(y)$ be the subspace of dimension $n-1$ of the tangent vector space which is perpendicular to y. The vectors of $T(y)$, for all $y \varepsilon B_{n1}$, form a bundle of complex vector spaces of dimension $n-1$ over B_{n1}. The product of this bundle and the bundle $p_{n1}: \tilde{B}_{n1} \to B_{n1}$ is clearly equivalent to the bundle over B_{n1} induced by the mapping $q_{n1}: B_{n1} \to M$. If we denote by

$$C(t) = \sum_{i=0}^{n} C_i t^i, \qquad C_0 = 1,$$

the characteristic polynomial in M, the characteristic polynomial of the induced bundle is

$$q_{n1}{}^*C(t) = \sum_{i=0}^{n} q_{n1}{}^*(C_i)t^i.$$

On the other hand, the characteristic polynomial of the bundle $p_{n1}: \tilde{B}_{n1} \to B_{n1}$ is $1 + u_1 t$, while that of the other factor is of the form

$$D(t) = \sum_{j=0}^{n-1} D_j t^j, \qquad D_0 = 1.$$

By the duality theorem we have therefore, identically in t,

$$\sum_{i=0}^{n} q_{n1}{}^*(C_i)t^i = (\sum_{i=0}^{n-1} D_i t^i)(1 + u_1 t).$$

Equating the corresponding coefficients of t^i, we get

$$q_{n1}{}^*(C_i) = D_i + u_1 D_{i-1}, \qquad i = 1, \cdots, n,$$

where we define $D_{-1} = D_n = 0$. Elimination of the D's gives formula (10). This proves Theorem 5.

In order to prove Theorem 6, we need the following algebraic lemma:

LEMMA 6.1. *Let*

$$C(t) = 1 + C_1 t + \cdots + C_n t^n \tag{40}$$

be a polynomial in t with coefficients in a commutative ring with unit element 1. To the elements C_i introduce $\bar{C}_k$, $k = 1, 2, \cdots$ as the coefficients of the formal power series

$$\bar{C}(t) = \sum_{k=0}^{\infty} \bar{C}_k t^k, \qquad \bar{C}_0 = 1, \tag{41}$$

so that

$$C(t)\bar{C}(t) = 1. \tag{42}$$

Let u be an element of the ring satisfying the equation

$$u^n C(-\frac{1}{u}) \equiv u^n(1 - \frac{C_1}{u} + \cdots + (-1)^n \frac{C_n}{u^n}) = 0. \tag{43}$$

Then we have, for any integer $N \geqq 1$,

$$u^{N+n-1} = (-1)^N \bar{C}_N u^{n-1} + \text{lower powers of } u. \tag{44}$$

To prove the lemma we write

$$\begin{aligned} 1 - C(t)(1 + \bar{C}_1 t + \cdots + \bar{C}_{N-1} t^{N-1}) &= C(t)(\bar{C}_N t^N + \cdots) \\ &= t^N D(t) + t^{n+N} P(t), \end{aligned}$$

where $P(t)$ is a power series in t and $D(t)$ a polynomial in t of degree $\leqq n-1$:

$$D(t) = D_0 + D_1 t + \cdots + D_{n-1} t^{n-1}.$$

Comparing the coefficients of t^N in both sides, we get

$$\bar{C}_N = D_0.$$

We now put $t = -1/u$ in the last identity and multiply the two sides by u^{n+N-1}. This gives

$$u^{n+N-1} - \{u^n C(-\frac{1}{u})\}\{u^{N-1} - \bar{C}_1 u^{N-2} + \cdots + (-1)^{N-1}\bar{C}_{N-1}\}$$

$$= (-1)^N u^{n-1} D(-\frac{1}{u}) + (-1)^{n+N}\frac{1}{u} P(-\frac{1}{u}).$$

This is a formal identity in the positive and negative powers of u. Restricting ourselves to the non-negative powers and taking account of the hypothesis of the Lemma, we get

$$u^{n+N-1} = (-1)^N u^{n-1} D(-\frac{1}{u}).$$

From this the Lemma follows.

We proceed now to the proof of Theorem 6. As in the proof of Theorem 5, we consider the bundle induced over B_{nr} by the mapping $q_{nr}: B_{nr} \to M$. Since M has an Hermitian metric, an element $L_0 \subset L_1 \subset \cdots \subset L_{r-1}$ of B_{nr} determines, and can be determined by, r mutually perpendicular directions $y_1, \cdots, y_r$ with the same origin, such that $y_1, \cdots, y_i$ span L_{i-1}, $i = 1, \cdots, r$. The induced bundle is therefore equivalent to the product of a bundle of vector spaces of dimension $n-r$ and the bundles of one-dimensional vector spaces over each of $y_1, \cdots, y_r$. We denote by

$$k(t) = k_0 + k_1 t + \cdots + k_{n-r} t^{n-r}, \qquad k_0 = 1, \tag{45}$$

and $1 + u_i t$, $i = 1, \cdots, r$, their respective characteristic polynomials. From the duality theorem we have

$$D(t) \equiv \sum_{i=0}^{n} D_i t^i = k(t) \prod_{j=1}^{r} (1 + u_j t), \tag{46}$$

where

$$D_i = q_{nr}^*(C_i), \qquad i = 0, \cdots, n. \tag{47}$$

It is easy to see that the classes u_i are related to the classes v_i by the relations

$$v_i = u_1 + \cdots + u_i, \qquad i = 1, \cdots, r, \tag{14}$$

as given in the Introduction. We proceed to derive from (46) a formula for the expression under the parenthesis in the left-hand side of (15). In what follows, multiplication of cohomology classes will be understood in the sense of cup product; it is commutative, because the classes concerned are all even-dimensional.

We introduce the polynomial

$$D'(t) = \sum_{i=0}^{n-r+1} D_i' t^i = (1 + u_r t) k(t), \tag{48}$$

and define the formal power series

$$\begin{aligned} \bar{D}'(t) &= \sum_{i=0}^{\infty} \bar{D}_i' t^i, & \bar{D}_0' &= 1, \\ \bar{D}(t) &= \sum_{i=0}^{\infty} \bar{D}_i t^i, & \bar{D}_0 &= 1, \end{aligned} \tag{49}$$

by means of the relations

$$D(t)\bar{D}(t) = 1, \qquad D'(t)\bar{D}'(t) = 1. \tag{50}$$

Then we have

$$D(t)\bar{D}'(t) = (1 + u_1 t) \cdots (1 + u_{r-1} t).$$

This equation completely determines $\bar{D}'(t)$. It follows by observation that

$$\bar{D}'(t) = \bar{D}(t)(1 + u_1 t) \cdots (1 + u_{r-1} t).$$

From the definition of $D'(t)$ we get

$$u_r^{n-r+1} D'(-\frac{1}{u_r}) = 0.$$

By Lemma 6.1, we have therefore

$$u_r^{2n-r} = (-1)^n \bar{D}'_n u_r^{n-r} + \text{terms in lower powers of } u_r.$$

But

$$\bar{D}'_n = u_1 \cdots u_{r-1} \bar{D}_{n-r+1} + \cdots,$$

where the unwritten terms are of degrees $< r - 1$ in $u_1, \cdots, u_{r-1}$. It follows that

$$\begin{aligned} &u_1^{n-2} \cdots u_{r-1}^{n-r} u_r^{2n-r} \\ &\qquad = (-1)^n \bar{D}_{n-r+1} u_1^{n-1} \cdots u_{r-1}^{n-r+1} u_r^{n-r} + \cdots. \end{aligned}$$

Under the homomorphism Ψ_{nr} the unwritten terms go to 0, while the first term goes to $(-1)^n \bar{C}_{n-r+1}$. This proves Theorem 6.

7. Application to algebraic geometry. We now sketch briefly the application of the above results to the case in which M is a non-singular algebraic variety of dimension n in a complex projective space of higher dimension. For this purpose we introduce another bundle over M. It is the bundle $[\tilde{B}_{nr}]$ obtained from $\tilde{B}_{nr}$ by adjoining to each fiber of $\tilde{B}_{nr}$, which is a Cartesian product of r complex lines with the origins deleted, the origin and a point at infinity. Its fiber is therefore a Cartesian product of r complex projective lines $L_1 \times \cdots \times L_r$. It is acted on, in an intransitive manner, by the general linear group $G(n)$, so that $[\tilde{B}_{nr}]$ is also an associated bundle of the principal bundle over M with $G(n)$ as structural group.

At this point we have to distinguish between the notion of an algebraic variety in the classical sense as one imbedded in the projective complex space and that of an abstract variety in the sense of Weil [16] defined by means of "overlapping neighborhoods." The variety $B_{n,n}{}^*$, being the principal bundle, is clearly an abstract variety. By the theory of fiber bundles in algebraic geometry, which is entirely analogous to the topological case, it follows that the bundles B_{nr}, $[\tilde{B}_{nr}]$, as associated bundles, are also abstract varieties and that the projections $q_{nr}: B_{nr} \to M$, $p_{nr}: \tilde{B}_{nr} \to B_{nr}$ are rational mappings.

On $[\tilde{B}_{nr}]$ there are sub-varieties V_{i0}, $V_{i\infty}$, $i = 1, \cdots, r$, defined by setting to zero or to ∞ the i-th coordinate in the fiber. Utilizing these sub-varieties, we can define the classes v_i in B_{nr} according to the following lemma, which was first given by Weil: [17]

LEMMA 7.1. *To an analytic cross-section f of the bundle $p_{nr}: \tilde{B}_{nr} \to B_{nr}$ the cycle*

$$(p_{nr})_*(f(B_{nr}) \cdot V_{i\infty} - f(B_{nr}) \cdot V_{i0})$$

is defined up to linear equivalence. Its homology class is $\mathfrak{D}v_i$.

To prove Theorem 7 we need also the following lemma:

LEMMA 7.2. *On a non-singular algebraic variety the intersection class of a finite number of divisor classes contains an algebraic cycle.*

For classical algebraic varieties this follows by induction from the following statement: If Z is an algebraic cycle on M and D is a divisor in M, there exists a divisor D' which is equivalent to D and which intersects Z properly.

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[16] Weil [15], Chapter VII.

[17] Weil [16].

In fact, let F be a hypersurface in the ambient projective space of M, whose intersection with M consists of D and a divisors D_1 which does not contain a given point of Z.[18] Then D_1 intersetcs Z properly. Let F_1 be a hypersurface, of the same degree as F, which intersects both M and Z properly. Set $D_2 = F_1 \cdot M$. Then the divisor $D' = D_2 - D_1$ is equivalent to D and intersects Z properly.

From these two lemmas the proof of Theorem 7 follows immediately. For $\mathfrak{D}v_1, \cdots, \mathfrak{D}v_r$, and hence $\mathfrak{D}u_1, \cdots, \mathfrak{D}u_r$, contain divisor classes. Therefore $(\mathfrak{D}u_1)^{n-2} \cdots (\mathfrak{D}u_{r-1})^{n-r}(\mathfrak{D}u_r)^{2n-r}$ contains an algebraic cycle. Its projection in M, which is a cycle belonging to the homology class $(-1)^n \bar{C}_{n-r+1}$, is an algebraic cycle. This proves Theorem 7.

To make our proof complete, one should prove Lemma 7.2 for abstract varieties. While this is "undoubtedly" true, no proof of it has been published. Alternately, it would presumably be easy to imbed the fiber bundles B_{nr}, $[\tilde{B}_{nr}]$ into a projective space, and then the theorem would again follow. These are questions of algebraic geometry which cannot be completely discussed without a lengthy introduction. We shall therefore not take them up in this paper.

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ON THE INVERSE OF THE PARABOLIC DIFFERENTIAL OPERATOR $\partial^2/\partial x^2 - \partial/\partial t$.*

By PHILIP HARTMAN and AUREL WINTNER.

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1. Let R denote a (sufficiently small) simply connected region in an (x, y)-plane, say the rectangle $R = R(1)$, where

$$R(c): \ (-1 \leqq x \leqq 1, 0 \leqq y \leqq c), \tag{1}$$

and let R^0 be the interior of $R(1)$,

$$R^0: \ (-1 < x < 1, 0 < y < 1). \tag{2}$$

If $u = u(x, y)$ is a (real-valued) function on R^0, put

$$\partial u = u_{xx} - u_y; \tag{3}$$

$$\Delta u = u_{xx} + u_{yy}, \tag{3 bis}$$

if the partial derivatives occurring in the respective operators ∂, Δ exist for $u = u(x, y)$. In contrast, denote by $\partial^* u = \partial^* u(x, y)$, $\Delta^* u = \Delta^* u(x, y)$ the operators

$$\partial^* u = \lim \partial_h u; \tag{4}$$

$$\Delta^* u = \lim \Delta_h u, \tag{4 bis}$$

if the respective limits (4), (4 bis), where $h \to 0$, exist for the quotients $\partial_h u$, $\Delta_h u$ defined by

$$\begin{aligned} h^2 \cdot \partial_h u &= u(x + h, y) + u(x - h, y) + u(x, y - h^2) - 3u(x, y) \\ &\equiv u(x + h, y) - 2u(x, y) + u(x - h, y) \\ &\qquad - \{u(x, y) - u(x, y - h^2)\} \end{aligned} \tag{5}$$

and

$$\begin{aligned} h^2 \Delta_h u = {} & u(x + h, y) + u(x, y + h) + u(x - h, y) \\ & + u(x, y - h) - 4u(x, y). \end{aligned} \tag{5 bis}$$

The expression (4 bis)-(5 bis), formed in analogy to the contracted second derivative (Riemann, Hölder) of a function of a single variable, was introduced by Zaremba [9] as an extension of (3 bis), whereas Gevrey's corresponding extension of (3) is not (4)-(5) but a limit which presupposes the existence of u_x; cf. [1], p. 363.

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The definition (4) of $\partial^* u$ is unsymmetric. A more natural (but apparently more restrictive) definition results if (5) is replaced by

$$\partial_h(u) = \{u(x+h, y) - 2u(x, y) + u(x-h, y)\}/h^2 \\ - \{u(x, y) - u(x, y-k)\}/k,$$

where $|k| = h^2 > 0$. It turns out that the limit (4) exists and is continuous when $\partial_h u$ is defined by (5) if and only if the same holds when $\partial_h u$ is defined in the more symmetric manner; cf. the proof of (i*) and (ii*) below.

Let $G = G(x, y; x', y')$ denote the function $\gamma(s, t)$ of

$$s = x - x' \quad \text{and} \quad t = y - y' \tag{6}$$

defined for $(s, t) \neq (0, 0)$ by

$$\gamma \equiv 0 \text{ if } t \leqq 0 \text{ and } \gamma = t^{-\frac{1}{2}} \exp(-s^2/4t) \text{ if } t > 0; \tag{7}$$

so that

$$G(x, y; x', y') = \gamma(x - x', y - y'). \tag{7 bis}$$

It is classical that (7) annihilates, for fixed (x, y) distinct from (x', y'), the adjoint $\partial^2/\partial x'^2 + \partial/\partial y'$ of $\partial = \partial^2/\partial x^2 - \partial/\partial y$ and that, correspondingly,

$$u(x, y) = -\tfrac{1}{2}\pi^{-\frac{1}{2}} \iint_R G(x, y; x', y') f(x', y')\, dx'\, dy' \tag{8}$$

represents *a* solution of $\partial u = f(x, y)$ on the interior (2) of (1) if $f(x, y)$ is a sufficiently smooth function on R (the factor $-\frac{1}{2}\pi^{-\frac{1}{2}}$ in (8) corresponds to the "infinity" of the "source function" γ). Thus (7 bis) and the logarithm of the distance between (x, y) and (x', y') play analogous parts for the respective inhomogeneous extensions,

$$\partial u = f(x, y); \tag{9}$$

$$\Delta u = f(x, y), \tag{9 bis}$$

of the equations of Fourier and of Laplace,

$$\partial u = 0; \tag{10}$$

$$\Delta u = 0. \tag{10 bis}$$

It is well-known that *if $f(x, y)$ is just continuous* on R (instead of satisfying something like a Hölder condition), then (9 bis) cannot be handled directly but must be replaced by the second of the equations

$$\partial^* u = f(x, y); \tag{11}$$

$$\Delta^* u = f(x, y), \tag{11 bis}$$

to which end the theory of (10 bis) must be extended to the second of the equations

$$\partial^* u = 0; \tag{12}$$

$$\Delta^* u = 0. \tag{12 bis}$$

It appears however that, while the relevant local theory of (9 bis)-(12 bis), as well as that of the second of the homogeneous equations

$$(13) \qquad \partial u + fu = 0; \qquad\qquad (13\text{ bis}) \qquad \Delta u + fu = 0,$$

is fully developed (Zaremba [9], Petrini [6]; cf. Wintner [8]), the results available for (9)-(13) in the literature are far from being of comparable sharpness. The Hölder criterion of E. E. Levi [5], p. 239, and its generalization by Gevrey [1], pp. 350-352, supply only sufficient (and, as will be seen below, surely not necessary) conditions for the existence of the relevant derivatives of the formal solution (8) of (9), in contrast to Petrini's complete results on the corresponding questions on the logarithmic potential of a continuous density $f(x, y)$. The object of this paper is to fill out these gaps in the theory of (9)-(13).

2. By a solution $u = u(x, y)$ of (10) on (2) will be meant any function for which the derivatives u_{xx}, u_y, those occurring in (3), exist and are equal at every point of (2). Correspondingly, by a solution of (12) on (2) will be meant a function $u(x, y)$ defined on (10) in such a way that the limit (4) of (5) exists at every point of (2) and vanishes identically. The "solutions" of (11), (12) or (13), where $f = f(x, y)$ is any given (usually continuous) function, are defined similarly.

If u is a solution of $\partial u = 0$, then u_y and u_x exist but u need not be continuous. In fact, there exist on R^0 solutions u of $\partial u = 0$ which are discontinuous at every point of a dense subset (even of the second category) of R^0; cf. [8], p. 732 and the corresponding function ϕ in [2], p. 371, top. On the other hand, if u is of class C^1 on R^0, then, while all that it is clear from $u_{xx} = u_y$ is that u_{xx} is continuous, it turns out (cf. [4], p. 2, where reference is given to an earlier paper of Holmgren) that u must be analytic in x for fixed y, and of class C^∞ (though not in general analytic) in x and y together (hence in y for fixed x). Actually, the same is true if nothing beyond the continuity (or the local boundedness) of u is assumed:

(i) *If $u(x, y)$ is a solution of $\partial u = 0$ on R^0 and if $u(x, y)$ is locally bounded and measurable (for example, if u is continuous on R^0), then u is of class C^1 on R^0 (hence of class C^∞ and, for fixed y, analytic in x).*

It was pointed out in [2], pp. 368-369, that, under the assumption that u is continuous on R^0, assertion (i) is between the lines of the classical literature. A verification of (i) under this condition was sketched in [2], pp. 368-369. This proof depended on a maximum principle. The proof

(although not the statement) of this principle was given by Gevrey [1], pp. 372-374. Actually, the proof of the required maximum principle does not depend on the continuity of u, but only on the continuity of u on each of the lines $x = \text{const.}$ and $y = \text{Const.}$ (and the local boundedness of u). Hence (i) follows by an obvious modification of the proof of (I) in [2].

The assertion (i) has the following extension:

(i*) *If $u(x, y)$ is a continuous solution of $\partial^* u = 0$ on R^0, then it is of class C^1 (hence, a solution of $\partial u = 0$) on R^0.*

The assumption of continuity on u can be replaced by assumptions of local boundedness and measurability.

Clearly, (i*) leads, via (i), to the following corollary:

(i* bis) *If $u = v(x, y)$ and $u = w(x, y)$ are two continuous solutions of $\partial^* u = f(x, y)$, then the function $u = v - w$ is of class C^∞ (which means that $v - w$ is a solution of $\partial u = 0$, rather than just of $\partial^* u = 0$).*

3. In view of (3) and (4)-(5), this corollary of (i*) implies an enumeration of all continuous solutions u of $\partial u = f$ in terms of one of them, say of $u = u_0$, *if* there exists such a u_0. But this assumption need not be satisfied, not even if f is continuous:

(ii) *There exist on R^0 continuous functions $f(x, y)$ corresponding to which $\partial u = f$ has no continuous solution $u(x, y)$ on any open subset of R^0.*

The same holds when (9) is replaced by (13):

(ii′) *The assertion of* (ii) *on the inhomogeneous equation $\partial u = f(x, y)$ holds for the homogeneous equation $\partial u + f(x, y)u = 0$ also, if the solution $u \equiv 0$ of the latter is disregarded.*

This has the following variant:

(ii″) *There exist on R positive, continuous functions f corresponding to which the differential equation $u_{xx} - f(x, y)u_y = 0$ fails to possess any continuous solution $u(x, y) \not\equiv \text{const.}$ on any subdomain of R.*

The point in the transition from the differential operator ∂ to its "Abelian" form ∂^* is not revealed by the above theorems. It is made clear however if (ii) is contrasted with the following existence theorem:

(ii*) *If $f(x, y)$ is continuous on the closure R of R^0, then $\partial^* u = f$ possesses on R^0 continuous solutions $u(x, y)$; in fact,* (8) *is such a solution* (whence *all* continuous solutions follow by (i* bis)).

4. The following theorems indicate the boundary between $\partial u = f$ and $\partial^* u = f$ more sharply than do (ii) and (ii*):

(iii) *If f is continuous on R, and if u is the particular solution* (8) *of $\partial^* u = f$ on R^0, then*

(iiia) *u is uniformly continuous on R^0 (that is, continuous on R);*

(iiiβ) *$u_y = \partial u/\partial y$ need not exist; although*

(iiiγ) *$D^{1-\epsilon}{}_y u = \partial^{1-\epsilon} u/\partial y^{1-\epsilon}$* (Liouville) *exists and is continuous on R^0 for every $\epsilon > 0$;*

(iiia) *$u_x = \partial u/\partial x$ exists and is continuous on R^0;*

(iiib) *$u_{xx} = \partial^2 u/\partial x^2$ need not exist; although*

(iiic) *$D^{1-\epsilon}{}_x u_x = \partial^{2-\epsilon} u/\partial u^{2-\epsilon}$ exists and is continuous on R_0 if $\epsilon > 0$.*

According to (iiia) and (iiiγ), both operators

$$\partial/\partial x \pm \partial^{\frac{1}{2}}/\partial y^{\frac{1}{2}} \tag{14}$$

are applicable to every u, which is of formal interest, since ∂ is the product of the two (formally commutable) operators (14); cf. (3).

That the function (8) has the properties (iiiα) and (iiia) is contained in E. E. Levi [5], pp. 229-235; cf. Gevrey [1], pp. 343-344. That (8) satisfies (iiiγ) and (iiic) follows from considerations of Gevrey [1], pp. 357-360, in view of the connection (Weyl [7]) between Hölder conditions and fractional differentiation. Hence only the equivalent parts (iiiβ), (iiib) of (iii) have to be proved.

In connection with (iiiβ) and (iiib), it is worth mentioning that, *at a given point (x, y), either both of the derivatives u_y, u_{xx} exist or neither of them does. Furthermore, when they exist at a given point, then $\partial u = f$.* This is implied by a result of Gevrey [1], p. 369; cf. (iv) below and its proof.

The analogue for (11) of Petrini's theorem for (11 bis) is as follows:

(iv) *Let $f(x, y)$ be continuous on the closure of R^0 and let $u(x, y)$ be a continuous solution of $\partial^* u = f$ on R^0. Then necessary and sufficient for the existence of $u_y(x, y)$ and/or $u_{xx}(x, y)$ at a point (x, y) of R^0 is the existence of*

$$\lim_{\epsilon \to 0} \iint_D f(x', y') G_y(x, y; x', y')\, dx' dy', \tag{15}$$

where $D = D(\epsilon)$ denotes either the set of points (x', y') of R for which

$y' < y - \epsilon$ or the subset of R outside the rectangle $|x' - x| < \epsilon^{\frac{1}{2}}$, $y > y' > y - \epsilon$. If the limit (15) *exists, then $\partial u = f$ holds at (x, y).*

Since the limit (15) exists when f is a constant, it follows that (15) exists if and only if

(15 bis) $L = \lim_{\epsilon \to 0} L_\epsilon$, where

$$L_\epsilon = -\tfrac{1}{2}\pi^{-1} \int\int_D \{f(x', y') - f(x, y)\} G_y(x, y; x', y') dx' dy',$$

does; in which case, at the point (x, y),

$$u_y = z_y f + L \quad \text{and} \quad u_{xx} = f + z_y f + L, \tag{16}$$

where $z = z(x, y)$ is the function (8) belonging to $f \equiv 1$.

It will be noted that, corresponding to Petrini's criteria for (3 bis), the necessary and sufficient criterion (15) supplied by (iv) deals with the question of existence, rather than with the continuity, of the partial derivatives occurring in the differential operator. It will however be clear from the proof of (iv) that, in order to assure the existence of continuous derivatives u_{xx}, u_y on (2), it is sufficient to require the satisfaction of (15) uniformly on every compact subset of (2). This leads to the following:

(iv bis) *In order that some and/or every continuous solution $u = u(x, y)$ of $\partial^* u = f$ on R^0 have continuous partial derivatives u_{xx}, u_y, it is sufficient that there exists a (continuous) function $\psi(s, t)$ defined for small $s \geqq 0$, $t \geqq 0$, satisfying*

$$\int_{+0} \int_{+0} \psi(s, t) |\gamma_t(s, t)| \, ds dt < \infty$$

and

$$|f(x, y) - f(x', y')| \leqq \psi(|x - x'|, |y - y'|).$$

5. *Proof of* (i*). The definition (4) of $\partial^* u$ and the first representation of $h^2 \partial_h u$ in (5) show that if $\partial^* u$ exists and is positive on some square S: $|x - x_0| \leqq \epsilon$, $|y - y_0| \leqq \epsilon$ contained in R^0, then the maximum of u on S is assumed on the lateral ($x = x_0 \pm \epsilon$) or the lower ($y = y_0 - \epsilon$) boundaries of S. Since u is continuous, there exists a (unique) continuous function $v = v(x, y)$ on S such that v (is of class C^∞ and) satisfies $\partial v = 0$ in the interior of S and $v \equiv u$ on the lateral and lower boundaries of S; cf., e. g., (IV) in [2], p. 369. There also exists a function $w = w(x, y)$ of

class C^∞ on S satisfying $\partial w = 1$, while $w = 0$ on the lateral and lower boundaries of S.

The proof of (i*) can now be completed by the arguments used by Zaremba [9], pp. 146-147, to pass from $\Delta^* u = 0$ to $\Delta u = 0$. Thus, if $\lambda > 0$ is an arbitrary constant, then either of the two functions $U = \pm (u - v) + \lambda w$ is continuous on S, vanishes on the lateral and lower boundaries of S and satisfies $\partial^* U = \lambda > 0$ on the interior of S. By the above maximum principle for ∂^*, it follows that $U \leqq 0$, that is, $| u - v | \leqq \lambda w$ on S. On letting $\lambda \to 0$, it is seen that $u \equiv v$ on S.

6. *Preliminary estimates.* In what follows, the abbreviation $G(x, y)$ will be used for $G(x, y; x', y')$, and ϵ will denote a (small) positive number.

LEMMA. *Let $I_k(\epsilon) = I_k(\epsilon; x, y)$, where $k = 1, 2, 3, 4$, denote the respective integrals*

$$(17_1) \qquad I_1(\epsilon) = \int_{y-\epsilon}^{y} \int_{-1}^{1} G(x, y)\, dx'dy',$$

$$(17_2) \qquad I_2(\epsilon) = \int_{0}^{y-\epsilon} \int_{-1}^{1} | G_{yy}(x, y)|\, dx'dy',$$

$$(17_3) \qquad I_3(\epsilon) = \int_{0}^{y-\epsilon} \int_{-1}^{1} | G_{xxx}(x, y)|\, dx'dy',$$

$$(17_4) \qquad I_4(\epsilon) = \int_{y-\epsilon}^{y} \int_{|x'-x| \geqq \delta} | G_y(x, y)|\, dx'dy', \qquad \text{where } \delta = \epsilon^{\frac{1}{3}} > 0.$$

Then, uniformly on every compact subset of R^0,

$$(18) \qquad \epsilon^{-1} I_1(\epsilon), \quad \epsilon I_2(\epsilon) \quad \epsilon^{\frac{1}{3}} I_3(\epsilon), \quad I_4(\epsilon) \text{ are } O(1) \text{ as } \epsilon \to 0.$$

It should be mentioned that the assertion of this Lemma remains valid if $\epsilon^{-1} I_1(\epsilon)$, $\epsilon I_2(\epsilon)$ in (18) are replaced by $\epsilon^{-1} I_1(\eta)$, $\epsilon I_2(\delta)$, respectively, where η, δ are functions of ϵ (and x, y) satisfying $0 \leqq \eta \leqq$ Const. ϵ, $0 <$ const. $\epsilon \leqq \delta \leqq y$. This is clear from the monotony of I_1, I_2 with respect to ϵ.

The proof of the Lemma will follow procedures used by Levi; cf. [5], pp. 231-232.

Proof of the Lemma. For $a > 0$, $\beta > 0$ and $b > a \geqq 0$, $c \geqq 0$, put

$$(19) \qquad I_{a\beta}(a, b; c) = \int_{a}^{b} t^{-\beta} \int_{c}^{\infty} s^a \exp(-s^2/4t)\, ds\, dt.$$

Then the change of variables $s/t^{\frac{1}{2}} = v$, $t = t$ transforms (19) into

$$(20)\qquad I_{\alpha\beta}(a, b; c) = \int_a^b t^{\frac{1}{2}\alpha-\beta+\frac{1}{2}} \int_{c(t)}^{\infty} v^\alpha \exp(-v^2/4)\, dv\, dt, \text{ where } c(t) = c/t^{\frac{1}{2}}.$$

Hence, if $\frac{1}{2}\alpha - \beta + 3/2 \neq 0$ and if $a > 0$ when $\frac{1}{2}\alpha - \beta + 3/2 < 0$, then

$$(21)\qquad I_{\alpha\beta}(a, b; c) \leqq I_{\alpha\beta}(a, b; 0) \leqq \text{Const.} \mid b^{\frac{1}{2}\alpha-\beta+3/2} - a^{\frac{1}{2}\alpha-\beta+3/2} \mid;$$

if $\frac{1}{2}\alpha - \beta + 3/2 = 0$ and $a > 0$, then

$$(22)\qquad I_{\alpha\beta}(a, b; 0) \leqq \text{Const.} \log b/a,$$

where the Const. in (21) and (22) depends only on α and β.

In view of (7) and (7 bis), the integrals (17_1)-(17_4) are majorized by 2 times

$$(23_1)\qquad \int_0^{\epsilon}\int_0^{\infty} \gamma(s, t)\, ds\, dt; \qquad (23_2)\qquad \int_{\epsilon}^{y}\int_0^{\infty} \mid \gamma_{tt}(s, t) \mid ds\, dt$$

$$(23_3)\qquad \int_{\epsilon}^{y}\int_0^{\infty} \mid \gamma_{sss}(s, t) \mid ds\, dt; \qquad (23_4)\qquad \int_0^{\epsilon}\int_{\delta}^{\infty} \mid \gamma_t(s, t) \mid ds\, dt,$$

respectively, where $y > \epsilon$. If $t > 0$, then (7) gives

$$(24)\qquad \gamma_t = \tfrac{1}{2}t^{-3/2}(\tfrac{1}{2}s^2/t - 1)\exp(-s^2/4t),$$

$$(25)\qquad 16\gamma_{tt} = (12t^{-5/2} - 12s^2t^{-7/2} + s^4t^{-9/2})\exp(-s^2/4t),$$

$$(26)\qquad 8\gamma_{sss} = 8\gamma_{ts} = (6st^{-5/2} - s^3t^{-7/2})\exp(-s^2/4t).$$

The integral in (23_1) is $I_{0\frac{1}{2}}(0, \epsilon; 0) \leqq \text{Const.}\, \epsilon$. By (25), the function (23_2) is majorized by a constant multiple of

$$I_{0\ 5/2}(\epsilon, y; 0) + I_{2\ 7/2}(\epsilon, y; 0) + I_{4\ 9/2}(\epsilon, y; 0) \leqq \text{Const.} \mid y^{-1} - \epsilon^{-1} \mid.$$

By (26), (23_3) is majorized by a constant multiple of

$$I_{1\ 5/2}(\epsilon, y; 0) + I_{3\ 7/2}(\epsilon, y; 0) \leqq \text{Const.} \mid y^{-\frac{1}{2}} - \epsilon^{-\frac{1}{2}} \mid.$$

This proves the statements concerning (17_1)-(17_3).

Since (24) shows that $\gamma_t > 0$ for $0 < t (< \frac{1}{2}\epsilon) \leqq \frac{1}{2}s^2$, the integral in (23_4) is majorized by the sum of

$$\int_{\delta}^{\infty}\int_0^{\frac{1}{2}\epsilon} \gamma_t\, ds\, dt < \int_0^{\infty} \gamma(s, \tfrac{1}{2}\epsilon)\, ds = 2^{\frac{1}{2}} \int_0^{\infty} \exp(-s^2/4)\, ds$$

and a constant multiple of

$$I_{2\,5/2}(\tfrac{1}{2}\epsilon, \epsilon; 0) + I_{0\,3/2}(\tfrac{1}{2}\epsilon, \epsilon; 0) \leqq \text{Const.} \log(\epsilon/\tfrac{1}{2}\epsilon).$$

This proves the assertion concerning (17_4).

7. *Proof of* (ii*). In order to prove that (8) satisfies $\partial^* u = f$ on R^0, it will first be shown that the "Lebesgue constant" $L(h) = L(h; x, y)$ defined by

$$L(h) = \int\int_R |\partial_h G(x, y)|\, dx'dy' \tag{27}$$

is bounded as $h \to 0$, if the point (x, y) of R^0 is fixed. Put

$$\epsilon = h^2 > 0, \qquad h > 0. \tag{28}$$

The contribution of the rectangle $|x'| < 1$, $y < y' < 1$ to (27) is 0. The contribution of the rectangle $|x'| < 1$, $y - 2\epsilon < y' < y$ is majorized by the sum of

$$\epsilon^{-1} \int_{y-2\epsilon}^{y} \int_{-1}^{1} |G(x+h, y) + G(x-h, y) - 3G(x, y)|\, dx'dy'$$

and

$$\epsilon^{-1} \int_{y-2\epsilon}^{y-\epsilon} \int_{-1}^{1} G(x, y-\epsilon)\, dx'dy'.$$

In view of the Lemma (cf. (17_1) and (18)), this contribution is $O(1)$ as $\epsilon \to 0$.

The integrand of (27) does not exceed the sum of the absolute values of

$$\epsilon^{-1}\{G(x+h, y) - 2G(x, y) + G(x-h, y)\} - G_{xx}(x, y) \tag{31}$$

and

$$G_y(x, y) - \epsilon^{-1}\{G(x, y) - G(x, y-\epsilon)\}, \tag{32}$$

since $G_{xx} - G_y = 0$. The expressions (31) and (32) can be written as

$$\epsilon^{-1} \int_0^h \int_{-w}^{w} \int_0^v G_{xxx}(x+u, y)\, du\, dv\, dw$$

and

$$\epsilon^{-1} \int_0^{\epsilon} \int_0^v G_{yy}(x, y-u)\, du\, dv, \tag{33}$$

respectively. Consequently, in order to prove the boundedness of (27), it is sufficient to prove the boundedness of

$$\epsilon^{-1}\int_0^h\int_{-w}^{w}\Big|\int_0^v\Big\{\int_0^{y-2\epsilon}\int_{-1}^{1}|G_{xxx}(x+u,y)|\,dx'dy'\Big\}du\Big|\,dv\,dw \tag{34}$$

and of

$$\epsilon^{-1}\int_0^{\epsilon}\int_0^{v}\Big\{\int_0^{y-2\epsilon}\int_{-1}^{1}|G_{yy}(x,y-u)|\,dx'dy'\Big\}du\,dv. \tag{35}$$

In view of the Lemma (cf. (17_3)), the interior integral $\{\cdots\}$ in (34) is $O(\epsilon^{-\frac{1}{2}})$ as $\epsilon\to 0$, uniformly in u. Hence (34) is

$$\epsilon^{-1}\int_0^h\int_{-w}^{w}\Big|\int_0^v O(\epsilon^{-\frac{1}{2}})du\Big|\,dv\,dw = O(\epsilon^{-1}h^3\epsilon^{-\frac{1}{2}}) = O(1), \tag{36}$$

since $h=\epsilon^{\frac{1}{2}}$. Similarly, the interior integral $\{\cdots\}$ in (35) is $I_2(2\epsilon;x,y-u)$, by (17_2), and is therefore $O(\epsilon^{-1})$ uniformly for $u(\leqq\epsilon)$, by the Lemma and the remark following it (where $\delta(2\epsilon;x,y-u)=2\epsilon-u\geqq\frac{1}{2}(2\epsilon)$). Thus (35) is $O(\epsilon^{-1}\epsilon^2\epsilon^{-1})=O(1)$.

This proves the boundedness of $L(h)$ as $h\to 0$. The assertion (11) can be deduced at once. For let (x_0,y_0) be a point of R^0 and let $\epsilon>0$. Choose $\delta=\delta_\epsilon>0$ so small that the square S: $|x-x_0|\leqq\delta$, $|y-y_0|\leqq\delta$ is in R^0 and that $|f(x,y)-f(x_0,y_0)|<\epsilon$ for any point (x,y) of S. Write $f=f_1+f_2$, where $f_1(x,y)$ is $f(x,y)$ or $f(x_0,y_0)$ according as (x,y) is not or is in S. Thus $|f_2(x,y)|<\epsilon$ at every point (x,y) of R. Let u_1 and u_2 denote the functions (8), when f is replaced by f_1 and f_2, respectively. (The discontinuities of f_1 and f_2 on the boundary of S do not affect the argument to follow.) Since f_1 is smooth (in fact, a constant on S), the function u_1 is of class C^∞ and satisfies $\partial u_1=f_1\equiv f(x_0,y_0)$ in the interior of S. In particular, $\partial^* u_1=f(x_0,y_0)$ in S. Since $|f_2|\leqq\epsilon$, it follows from $L(h)=O(1)$ that, as $h\to 0$, $\limsup|\partial_h u_2|\leqq$ Const. ϵ for all (x,y) in R^0. Hence, if $(x,y)=(x_0,y_0)$, then $\limsup|\partial_h u-f(x_0,y_0)|\leqq$ Const. ϵ, where $h\to 0$. Since $\epsilon>0$ is arbitrary, this proves that the limit $\partial^* u(x_0,y_0)$ exists and is $f(x_0,y_0)$, as claimed by (ii*).

8. *Proof of* (iv). The assertions (i* bis) and (ii*) imply that it is sufficient to consider the particular solution (8) of $\partial^* u=f$ in the proof of (iv). It will be shown that, if (x,y) is a point of R^0, the difference between the integral in (15) and the ratio $-2\pi^{\frac{1}{2}}\{u(x,y)-u(x,y-h)\}/h$ tends

to 0 as $\epsilon = 2 \mid h \mid \to 0$. To this end, the corresponding "Lebesgue constant" will be proved to be bounded.

Let

$$\epsilon = 2 \mid h \mid, \quad \text{where} \quad h \gtrless 0, \tag{37}$$

and put

$$L_1(h) = \int_0^{y-2|h|} \int_{-1}^{1} \mid G_y(x, y) - h^{-1}\{G(x, y) - G(x, y-h)\} \mid dx'dy'. \tag{38}$$

Thus, if $D(\epsilon) = R(y - \epsilon)$, the Lebesgue constant of the problem is majorized by

$$L_1(h) + 2 \mid h \mid^{-1} I_1(2 \mid h \mid; x, y) + 2 \mid h \mid^{-1} I_1(2 \mid h \mid - h; x, y - h);$$

cf. (17_1). If the region $D(\epsilon)$ is the portion of $R(y)$ outside the rectangle $\mid x' - x \mid < \epsilon^{\frac{1}{3}}$, $y > y' > y - \epsilon$, then (17_4) must be added to this majorant.

In view of the Lemma, the boundedness of the Lebesgue constant follows if it is shown that (38) is bounded as $h \to 0$. This has been proved in Section 7 when $h > 0$; cf. the considerations concerning (32), (33) and (35). The case $h < 0$ can be treated in the same manner, since the proof of the boundedness of (35) depended on the inequality $u \leqq \epsilon$ (which is satisfied if $u \leqq 0$).

Let (x, y) be fixed and let $\epsilon > 0$ be so small that the points (x', y') of the rectangle S: $\mid x' - x \mid \leqq \frac{1}{2}\epsilon^{\frac{1}{3}}$, $\mid y' - y \mid \leqq \epsilon$ are in R^0. Let f be written as $f = f_1 + f_2$, where $f_1 = f_1(x', y')$ is $f(x', y')$ or $f(x, y)$ according as the point (x', y') of R is not or is in S (so that f_1 is constant on S). Let u_1, u_2 denote the functions (8) which result if f is replaced by f_1, f_2, respectively.

Clearly, u_1 is of class C^∞ on the interior of S; in particular $\partial u_1/\partial y$ and $\partial^2 u_1/\partial x^2$ are given by the formulae (16), where L is the limit in (15 bis) in which f is replaced by f_1; cf. [5], p. 238. Since $f_1(x', y') - f_1(x, y) = 0$ if (x', y') is on S, the corresponding L is L_ϵ, where L_ϵ is defined as in (15 bis). Thus, $\lim \partial u_1/\partial y$ and/or $\lim \partial^2 u_1/\partial x^2$, where $\epsilon \to 0$, exists if and only if the limit in (15 bis) does.

In view of the boundedness of the "Lebesgue constants," the value of

$$\limsup \mid \{u_2(x, y + h) - u_2(x, y)\}/h \mid \qquad (h \to 0)$$

does not exceed const. η, where η is the maximum of $\mid f_2 \mid$ on R. Hence

$$\limsup \mid \{u(x, y + h) - u(x, y)\}/h - \partial u_1/\partial y \mid \leqq \text{const. } \eta, \text{ where } h \to 0.$$

The continuity of f implies that $\eta = \eta(\epsilon) \to 0$ as $\epsilon \to 0$. Consequently,

$u_y(x, y)$ exists if and only if $\lim \partial u_1/\partial y$, where $\epsilon \to 0$, does (that is, if and only if the limit in (15 bis) does). This proves the first part of (iv) (and the formulae (16) as well, if (15) and/or the limit in (15 bis) exists). The last part of (iv) is a consequence of (16).

Remark. The proof of the boundedness of the "Lebesgue constants" shows that if u is the function (8) or any continuous solution of (11), where f is continuous, then *any one of the following conditions is necessary and sufficient for the existence of* $u_{xx}(x, y)$ *and/or* $u_y(x, y)$ *at the point* (x, y) *of* R^0:

(I) $\lim (u(x, y) - u(x, y - h))/h$ exists as $h \to +0$;

(II) $\lim (u(x, y) - u(x, y - h))/h$ exists as $h \to -0$;

(III) $\lim (u(x + h, y) - 2u(x, y) + u(x - h, y))/h^2$ exists as $h \to 0$.

9. *Proof of* (iiib) *and* (iiiβ). An example of a continuous function $f(x, y)$ for which (8) fails to possess the partial derivative u_y at a point of R^0 can now be given at once. Define an auxiliary function $f^* = f^*(s, t)$ as follows: For all s, put

$$(39) \qquad f^* = 0 \text{ if } t \leqq 0, \; f^* = (1 - \tfrac{1}{2}s^2/t)/\{(1 + \tfrac{1}{2}s^2/t)\log t\} \text{ if } t > 0.$$

Clearly, f^* is continuous. Let (x_0, y_0) be a point of R^0 and put

$$(40) \qquad f(x, y) = f^*(x_0 - x, y_0 - y).$$

It will be verified that, for the choice (40) of f, the limit (15) does not exist at $(x, y) = (x_0, y_0)$.

The integral in (15), where $D(\epsilon) = R(y - \epsilon)$, becomes

$$\int_{\epsilon}^{y} \int_{-1}^{1} f^*(s, t)\gamma_t(s, t)\, ds\, dt$$

under the substitution $s = x - x'$, $t = y - y'$. If the integration variables are changed by the transformation $v = s/t^{\frac{1}{2}}$, $t = t$, then (24) and (39) show that the last integral becomes

$$\int\int \tfrac{1}{2} \mid t \log t \mid^{-1} (1 - \tfrac{1}{2}v^2)^2 (1 + \tfrac{1}{2}v^2)^{-1} \exp(-v^2/4)\, dv\, dt,$$

where the domain of integration is $\mid v \mid \leqq t^{-\frac{1}{2}}$, $\epsilon \leqq t \leqq y$. The integrand is non-negative and so the last integral is obviously minorized by

$$\text{const.} \int_{\varepsilon}^{y} |\, t \log t \,|^{-1}\, dt, \text{ where const.} > 0.$$

This proves (iiib) and (iiiβ).

10. *The assertions* (ii′) *and* (ii″). The proofs of these assertions will be omitted. For (ii′) can be proved by the procedure employed in [8] dealing with the equation $\Delta u = f$, except that, in the arguments of [8], Petrini's criterion [6], pp. 131-134, is replaced by assertion (iv). A corresponding remark applies to Theorem (ii″), which can be proved by adopting the method of [3], pp. 267-269, which is an elaboration of the one in [8].

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ON ELLIPTIC MONGE-AMPÈRE EQUATIONS.*

By PHILIP HARTMAN and AUREL WINTNER.

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1. *Introduction.* The partial differential equation to be considered will not be the general elliptic Monge-Ampère equation but that of the particular form

$$rt - s^2 = f(x, y, z, p, q), \tag{1}$$

where

$$f > 0 \tag{2}$$

(so that the linear terms in r, s, t of the general Monge-Ampère equation are missing).

A classical theorem of S. Bernstein states that a solution $z = z(x, y)$ of class C^3 of an analytic elliptic partial differential equation of second order is analytic. According to Lichtenstein [9], pp. 935-936, the assumption that z is of class C^3 can be lightened to the hypothesis that z is of class C^2, if the differential equation is linear in the second order partial derivatives. It will be shown below that the same is true when the differential equation is of the type (1). The proof will be based on standard techniques in the theory of elliptic partial differential equations and on a manifestation of a general principle, according to which a function $z(x, y)$ cannot be smoother as a function of the variables $(u, v) = (x, q(x, y))$ than is expected a priori.

For other manifestations of the latter principle, see [4], pp. 133-134, and for those of a similar principle applied to the variables $(u, v) = (p(x, y), q(x, y))$, see [5], p. 306 and [6]. Along the lines of the above-mentioned principle, the Legendre-like transformation $(x, y) \to (x, q)$ (cf. [1], p. 140) was applied to $rt - s^2$ in [3], pp. 152-153, in another connection.

The theorem announced above is as follows:

(i) *If $f(x, y, z, p, q)$ is a positive analytic function of its arguments on some five-dimensional domain and if $z = z(x, y)$ is a solution of class C^2 of* (1) *on some (x, y)-domain, then z is analytic.*

This theorem (i) is known in the particular cases where f in (1) depends

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only on (x, y) and not on z, p, q (Pogoreloff [11]); so that (1) is of the form

$$rt - s^2 = \phi(x, y) \qquad (\phi > 0) \tag{3}$$

(in which case the assumption on the solution z can be further lightened). (i) and the results to be obtained below will depend on the consideration of equations of the type (3) in which it is assumed that ϕ is of class C^1 or, in fact, subject only to a Hölder condition. It may be mentioned in this connection that Pogoreloff's results [11] cannot be extended from (3) to (1), since his method involves the assumption that ϕ in (3) is of class C^3.

If $0 < \lambda < 1$, a function $\phi(x, y)$ will be called of class $C^0(\lambda)$ on the circle

$$D\colon\ x^2 + y^2 < 1 \tag{4}$$

if, for every positive $r < 1$, there exists a constant $M = M(r)$ satisfying

$$|\,\phi(x, y) - \phi(x', y')\,| \leqq M(|\,x - x'\,|^\lambda + |\,y - y'\,|^\lambda) \tag{5}$$

for every pair of points (x, y), (x', y') of the circle

$$D_r\colon\ x^2 + y^2 < r^2 \qquad (r < 1;\ D_1 = D). \tag{6}$$

If $n = 1, 2, \cdots$ and $0 < \lambda < 1$, a function $\phi(x, y)$ will be called of class $C^n(\lambda)$ on D if it possesses all partial derivatives of the n-th order and these are of class $C^0(\lambda)$ on D. In this terminology, the main result of the present paper can be stated as follows:

(I_n) *Let $\phi(x, y)$ be positive and of class $C^n(\lambda)$ on D for a fixed $n \geqq 0$, and let $z = z(x, y)$ be a solution of class C^2 of (3) on D. Then $z(x, y)$ is of class $C^{n+2}(\mu)$ on D for every $\mu < \lambda$.*

These assertions, where $n = 0, 1, \cdots$, are analogues of standard theorems dealing with solutions of the Poisson equation

$$r + t = \phi(x, y). \tag{7}$$

In addition to their use in the proof of (i), the assertions (I_n) have applications in differential geometry. For example, it will be shown that (I_n) implies the following theorem:

(ii_n) *Let $S\colon\ z = z(x, y)$ be a surface of class C^2 possessing a positive Gaussian curvature $K(x, y) = (rt - s^2)/(1 + p^2 + q^2)^2$ of class C^n. Then the surface S is of class C^{n+1}.*

The assumption that K is of class C^n can be lightened to the hypothesis that K is of class $C^{n-1}(\lambda)$ and, under this lightened hypothesis, it can be asserted that S is of class $C^{n+1}(\mu)$ for every positive $\mu < \lambda$.

If the assumption of the existence of a positive Gaussian curvature K of class C^n is replaced by the assumption of the existence of a mean curvature H of class C^n, then the corresponding assertion is known; cf. [4], Theorem (i), p. 127. On the other hand, (ii_n) seems to be new for two reasons; first, that it allows $n = 2$ and, second, that S is assumed to be of class C^2 only (rather than of class C^3, even when $n > 2$); cf. [4], Theorem (ii), p. 127. Thus (ii_2) answers in the positive the question formulated in [4], p. 128, since (ii_2) means that not only the first but also *the second of the theorems of* [4], p. 127, *remains true for* $n = 2$.

Pogoreloff [11], p. 88, has a result analogous to (ii_n). It states that if a convex surface S: $z = z(x, y)$ has a positive Gaussian curvature K which, *as a function of the normal vector,* is of class C^n, where $n \geqq 3$, then S is of class C^{n+1}. If the assumption that S is convex is strengthened to the assumption that S is of class C^2, then the corresponding assertion is contained in (ii_{n-1}); cf. the proof of (ii_n) below. Thus the "$n \geqq 3$" in this version of Pogoreloff's theorem can be improved to "$n \geqq 2$" (the case $n = 1$ being correct, but trivial). On the other hand, the theorem of Pogoreloff does not seem to contain (ii_n) even for large n, since his assumption requires that K be smooth as a function of the normal vector (cf. [5], p. 306), that is, of (p, q), rather than, as in (ii_n), of (x, y).

2. *An "integrated" form of* (3). If $z = z(x, y)$ is of class C^2 on D, then the form of Green's theorem given by the Lemma of [2], p. 761, shows that

$$(8) \qquad \int_J p(sdx + tdy) = \iint_B (rt - s^2)dxdy = -\int_J q(rdx + sdy),$$

for every domain B bounded by a piecewise smooth Jordan curve J contained in D. Hence (3) can be writen in the form

$$(9) \qquad \int_J (ps - qr)dx + (pt - qs)dy = 2\iint_B \phi(x, y)dxdy.$$

It is clear from (3) that

$$(10) \qquad t = t(x, y) \neq 0$$

on D. Hence the transformation $(x, y) \to (u, v)$ defined by

$$u = x, \qquad v = q(x, y), \tag{11}$$

a transformation of class C^1 having the Jacobian

$$\partial(u, v)/\partial(x, y) = t(x, y), \tag{12}$$

is one-to-one if (x, y) is restricted to a small vicinity of a point of D, say to D_ϵ, where $\epsilon > 0$ is sufficiently small. Let D^* denote the (u, v)-image of D_ϵ under the transformation (11).

The equations (11) imply that

$$du = dx, \quad dv = sdx + tdy \quad \text{or} \quad dx = du, \quad dy = (dv - sdu)/t; \tag{13}$$

hence

$$z_u = p - qs/t; \tag{14_1}$$

$$z_v = q/t. \tag{14_2}$$

It is easily verified that (13) and (14) give

$$(ps - qr)dx + (pt - qs)dy = -(rt - s^2)z_v du + z_u dv. \tag{15}$$

Thus, by (3) and (12), the relation (9) can be written as

$$\int_J \phi z_v du - z_u dv = -2 \iint_B \phi/t \, dudv, \tag{16}$$

where B is the interior of any piecewise smooth Jordan curve J contained in D^*. The transformation of (3) into (16) was the object of this section.

3. *Functions $\phi(x, y)$ of class C^1.* Under the assumption that ϕ (> 0) is of class C^1, the relation (16), which is an identity in B, implies, by the Lemma of [2], p. 761, that

$$\int_J \phi^{\frac{1}{2}} z_v du - \phi^{-\frac{1}{2}} z_u dv = \iint_B \psi \, dudv, \tag{17}$$

where

$$\psi = \psi(u, v) = -2\phi^{\frac{1}{2}}/t - \tfrac{1}{2}\phi^{-3/2}(z_u \phi_u + \phi z_v \phi_v). \tag{18}$$

Consider the system of partial differential equations

$$a_u = \phi^{\frac{1}{2}} \beta_v, \qquad a_v = -\phi^{-\frac{1}{2}} \beta_u. \tag{19}$$

Since ϕ is positive and of class C^1, the coefficients $\phi^{\frac{1}{2}}$, $-\phi^{-\frac{1}{2}}$ in (19) are of

class C^1 on D^*. It follows therefore from a theorem of Lichtenstein [10] that (19) has a solution

$$\alpha = \alpha(u, v), \qquad \beta = \beta(u, v) \tag{20}$$

which is of class $C^1(\mu)$ for every $\mu < 1$. Furthermore, (20) possesses a positive Jacobian

$$\chi = \partial(\alpha, \beta)/\partial(u, v) \tag{21}$$

and defines a one-to-one mapping of D^* onto some (α, β)-domain D^{**}. It should be remarked, for later reference, that if ϕ, as a function of (u, v), is of class $C^n(\lambda)$, then (20) is of class $C^{n+1}(\mu)$ for every $\mu < \lambda$; cf. [10].

According to (21), the formulation (17) of (3) is transformed by (20) into

$$\int_J z_\beta d\alpha - z_\alpha d\beta = \iint_B \psi/\chi \, d\alpha d\beta. \tag{22}$$

Formally, this is equivalent to the Poisson equation

$$z_{\alpha\alpha} + z_{\beta\beta} = -\psi/\chi. \tag{23}$$

4. *Proof of* (I_n) *when* $n > 0$. Let $z = z(x, y)$ be a solution of class C^2 of (3). It is sufficient to prove that z has the asserted properties of smoothness in a vicinity D_ϵ of $(x, y) = (0, 0)$. It can also be supposed that

$$q = z_y(x, y) \neq 0 \tag{24}$$

at $(0, 0)$, hence on D_ϵ for sufficiently small $\epsilon > 0$; for otherwise $z(x, y)$ could be replaced by $z(x, y) + \text{const.}\, y$.

If ϕ is of class $C^n(\lambda)$, where $n \geqq 1$ and $0 < \lambda < 1$, then ϕ is of class C^1 and, on D_ϵ, the differential equation (3) can be written in the form (22) after the transformations $(x, y) \to (u, v) \to (\alpha, \beta)$, given by (11) and (20).

Since ψ/χ in the double integral (22) is a continuous function of (α, β), it follows from (22) that z as a function of (α, β) is of class $C^1(\mu)$ for every positive $\mu < 1$ (see [8], pp. 96-100); in fact, z as a function of (α, β) differs only by an harmonic function from the logarithmic potential having a density proportional to ψ/χ. Since (20) is of class $C^1(\mu)$ for every $\mu < 1$, it follows that z as a function of (u, v) is of class $C^1(\mu)$ for every $\mu < \lambda$ (< 1). Also, p, q are of class C^1 in terms of (u, v), since they are of class C^1 in (x, y). Hence (14_2) shows that t as a function of (u, v) (hence of (x, y)) is of

class $C^0(\mu)$, where $0 < \mu < \lambda$. Hence (14_1) and (24) show that the same is true of s as a function of (u, v) (hence of (x, y)). Finally, (3) and (10) show that the same is true of r as a function of (u, v) (hence of (x, y)). Consequently, $z = z(x, y)$ is a function of class $C^2(\mu)$ for every $\mu < \lambda$.

Let $n > 0$ be fixed. The assertion (II_n) will be proved by induction. Let $0 \leqq k < n$ and let it be assumed that $z = z(x, y)$ is of class $C^{k+2}(\mu)$ for every $\mu < \lambda$. It will be shown that z is of class $C^{k+3}(\mu)$ for every $\mu < \lambda$.

The transformation (11) is of class $C^{k+1}(\mu)$. Thus z, z_u, z_v, t, as functions of (u, v), are of class $C^{k+1}(\mu)$, $C^k(\mu)$, $C^k(\mu)$, $C^k(\mu)$, respectively. Hence (18) is of class $C^k(\mu)$.

Since $k + 1 \leqq n$ and the transformation (11) is of class $C^{k+1}(\mu)$ for every $\mu < \lambda$, it follows that ϕ as a function of (u, v) on D^* is of class $C^{k+1}(\mu)$ for every $\mu < \lambda$. Consequently, the transformation (20) is of class $C^{k+2}(\mu)$ and so its Jacobian χ is of class $C^{k+1}(\mu)$ for every $\mu < \lambda$; cf. the remark following (21). Since (18) is of class $C^k(\mu)$, the function ψ/χ on the right of (22) is of class $C^k(\mu)$ as a function of (α, β) or of (u, v).

By known facts on Poisson's equation (23) or its integrated form (22) (cf., e. g., [8]), it follows from (22) that z is a function of class $C^{k+2}(\mu)$ for every $\mu < \lambda$ as a function of (α, β). Since the transformation (20) is of class $C^{k+2}(\mu)$, it is seen that z is of class $C^{k+2}(\mu)$ as a function of (u, v).

Hence z_u, z_v (as well as p, q) are of class $C^{k+1}(\mu)$ as functions of (u, v). It follows from (14_1), (14_2), (3), respectively, that t, s, r are of class $C^{k+1}(\mu)$ as functions of (u, v), hence also as functions of (x, y).

Since the second order partial derivatives of z are of class $C^{k+1}(\mu)$, it follows that z is of class $C^{k+3}(\mu)$ for every $\mu < \lambda$. This completes the proof of (I_n) for $n > 0$.

5. *Proof of* (i). Let f and z satisfy the conditions of (i). Put

$$\phi(x, y) = f(x, y, z(x, y), p(x, y), q(x, y)).$$

Then $\phi(x, y)$ is of class C^1 and $z(x, y)$ satisfies (3). It follows from Section **3** and the first part of Section **4**, that $z(x, y)$ is of class $C^2(\mu)$ for every $\mu < 1$.

Hence ϕ is of class $C^1(\mu)$ for every $\mu < 1$. It follows therefore from (I_1) that z is of class $C^3(\mu)$ for every $\mu < 1$. (A simple induction shows that z is of class C^∞.) Hence the assertion (i) follows from Bernstein's theorem.

6. *Proof of* (I_0). Under the assumptions of (I_0), the function $\phi(x, y)$ cannot be assumed to be of class C^1, so that the reduction of (16) to (17) (and then to (22)) is not valid. The assertion of (I_0) will be proved by a device used by E. Hopf [7] to show that the C^3-assumption in Bernstein's theorem can be replaced by a $C^2(\lambda)$-assumption.

In view of (16) and the arguments of Section **4** (used to pass from the $C^k(\mu)$-character of z as a function of (u, v) to its $C^{k+1}(\mu)$-character as a function of (x, y)), the assertion (I_0) will be proved if the following theorem is verified:

If $z(u, v)$, $\phi(u, v) > 0$, $\psi(u, v)$ are functions of class C^1, $C^0(\lambda)$, C^0 (= continuous), respectively, on some simply connected (u, v)-domain D^* and if

$$\int_J \phi z_v du - z_u dv = \iint_B \psi \, du dv \tag{25}$$

holds for every domain B bounded by a piecewise smooth Jordan curve J contained in D^*, then $z(u, v)$ is of class $C^1(\mu)$ on D^* for every $\mu < \lambda$.

It is sufficient to show that $z(u, v)$ is of class $C^1(\mu)$ on every circle contained in D^*. Also, it can be supposed that the closure of the circle D: $u^2 + v^2 < 1$ is contained in D^*, and it is sufficient to show that z is of class $C^1(\mu)$ on D.

Let (u_0, v_0) be a point of D and put

$$\phi_0 = \phi(u_0, v_0). \tag{26}$$

Then (25) can be written as

$$\int_J \phi_0 z_v du - z_u dv = \int_J (\phi_0 - \psi) z_v du + \iint_B \psi \, du dv. \tag{27}$$

Also

$$\int_J z_u du + z_v dv = 0 \tag{28}$$

for every J. Introducing complex notation, put

$$w = w(u, v) = \phi_0^{\frac{1}{2}} u + iv \tag{29}$$

and

$$\omega = \omega(u, v) = \phi_0^{\frac{1}{2}} z_v + i z_u. \tag{30}$$

Then, if (28) is multiplied by $i\phi_0^{\frac{1}{2}}$ and added to (27), there results the relation

$$\int_J \omega dw = \int_J (\phi_0 - \phi) z_v du + \iint_B \psi \, du dv. \tag{31}$$

The Lemma in [2], p. 761, implies that if $G = G(u, v)$ is of class C^1 on a domain containing J, then

$$\int_J G\omega dw = \int_J G(\phi_0 - \phi) z_v du + \iint_B \{G\psi + G_v(\phi_0 - \phi) z_v + \omega(iG_u - \phi_0^{\frac{1}{2}} G_v)\} du dv.$$

Let $w_0 = \phi_0^{\frac{1}{2}} u_0 + i v_0$ and put

$$G = G(u, v) = G(u, v; u_0, v_0) = (w - w_0)^{-1}; \tag{32}$$

so that $iG_u - \phi_0^{\frac{1}{2}} G_v \equiv 0$ for $| w - w_0 | \neq 0$. Thus

$$\int_J G\omega dw = \int_J G(\phi_0 - \phi) z_v du + \iint_B (G\psi + G_v(\phi_0 - \phi) z_v) du dv, \tag{33}$$

if $B = B(\epsilon)$ is the domain bounded by the circles $u^2 + v^2 = 1$ and $| w - w_0 |^2 \equiv \phi_0 (u - u_0)^2 + (v - v_0)^2 = \epsilon^2$, where $\epsilon > 0$ is small.

As $\epsilon \to 0$, the contribution of the line integral around $| w - w_0 | = \epsilon$ to the left side of (33) tends to $-2\pi i \omega(w_0)$, while the contribution of the corresponding line integral on the right side of (33) tends to 0, since, according to (26),

$$\phi_0 - \phi(u, v) = O(| u - u_0 |^\lambda + | v - v_0 |^\lambda). \tag{34}$$

But (34) also shows that, as $\epsilon \to 0$, the double integral over $B = B(\epsilon)$ in (33) tends to the corresponding (absolutely convergent) integral over D: $u^2 + v^2 \leqq 1$. Hence

$$2\pi i \omega(w_0) = \int_C G\omega dw - \int_C G(\phi_0 - \phi) z_v du - \iint_D (G\psi + G_v(\phi_0 - \phi) z_v) du dv, \tag{35}$$

where D: $u^2 + v^2 < 1$ and C: $u^2 + v^2 = 1$.

In view of (26) and (30),

$$\omega(w_0) = \phi^{\frac{1}{2}}(u_0, v_0) z_v(u_0, v_0) + i z_u(u_0, v_0). \tag{36}$$

Hence it is sufficient to show that $\omega(w_0) = \omega(u_0, v_0)$ is of class $C^0(\mu)$ on $u_0^2 + v_0^2 < 1$, for every $\mu < \lambda$.

Since $G = \{\phi_0^{\frac{1}{2}}(u_0, v_0)(u - u_0) + i(v - v_0)\}^{-1}$, by (36), it is clear that the functions

$$\int_C G\omega dw, \qquad \int_C G(\phi_0 - \phi) z_v du, \qquad \iint_D G\psi \, dudv$$

of (u_0, v_0) are of class $C^0(\lambda)$ on $u_0^2 + v_0^2 < 1$. Thus in order to prove the case $n = 0$ of (I_n), all that remains to be verified is that

$$\iint_D G_v(\phi_0 - \phi) z_v dudv,$$

as a function of (u_0, v_0), is of class $C^0(\mu)$ on $u_0^2 + v_0^2 < 1$ for every $\mu < \lambda$. But this fact is contained in [7], Hilfssatz 3, pp. 203-204, since it is clear that the kernel $K(X; Z) = K(x_1, x_2; z_1, z_2) = (\phi^{\frac{1}{2}}(x_1, x_2) z_1 + i z_2)^{-1}$ satisfies the conditions of that lemma and that $G = K(X; Y - X)$ if $X = (u_0, v_0)$ and $Y = (u, v)$.

7. *Proof of* (ii_n). Let (ii_n^*) denote the strengthened form of (ii_n), as formulated after the statement of (ii_n). Thus (ii_n^*) claims that if $z = z(x, y)$ is of class C^2 and if the Gaussian curvature $K(x, y)$ is of class $C^n(\lambda)$, where $n = 0, 1, \cdots$ and $0 < \lambda < 1$, then z is of class $C^{n+2}(\mu)$ for every $\mu < \lambda$.

The function $z = z(x, y)$ satisfies (3), where

$$\phi(x, y) = K(x, y)(1 + p^2(x, y) + q^2(x, y))^2. \tag{37}$$

If K is of class $C^0(\lambda)$, then (37) is also. Hence (ii_0^*) follows from (I_0).

Let $n > 0$ and assume the truth of $(\mathrm{ii}_0^*), \cdots, (\mathrm{ii}_{n-1}^*)$. If K is of class $C^n(\lambda)$, it follows from (ii_{n-1}^*) that (37) is of class $C^{n+1}(\mu)$ for every $\mu < \lambda$. Hence (ii_n^*) follows from (I_n).

It is worth mentioning that the proof of (ii_n^*), where $n \geqq 1 \neq 0$, depends only on Sections **3** and **4**, and not on the device used in Section **6** in the proof of (I_0).

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SIMPLE DIFFERENTIALS OF SECOND KIND ON HODGE MANIFOLDS.*

By MAXWELL ROSENLICHT.

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1. Let V be a complex analytic manifold. A *simple differential* on V is a meromorphic exterior differential form of degree one. The simple differential ω is said to be *of the second kind* if, for each $p \in V$, there exists an open neighborhood U_p of p and a function f_p, meromorphic in U_p, such that $\omega = df_p$ in U_p. A differential of the first kind (one that is closed and everywhere holomorphic) is clearly of the second kind. Similarly, if F is any function meromorphic on all of V, then the differential $\omega = dF$ is of the second kind; such a differential is said to be *exact*.

Let V have complex dimension n and let ω be a simple differential of second kind on V. The polar locus of ω (i. e., the set of points at which ω is not holomorphic) is an analytic subvariety of V of dimension $n-1$ (or empty). If γ is any path on V whose endpoints are not in this polar locus then one can define the integral $\int_\gamma \omega$, which is a complex number depending only on ω, the endpoints of γ, and the homotopy class of γ among paths having these endpoints fixed. In particular, if γ is any 1-cycle, the period $\int_\gamma \omega$ depends only on the homology class of γ. If $\int_\gamma \omega = 0$ for each cycle γ, then we can write $\omega = dF$, where F is meromorphic on all of V. It follows that if Ω_2 denotes the vector space (over the complex number field C) of simple differentials of second kind on V, Ω_e the subspace of exact differentials, and B_1 the first Betti number of V, then

$$\dim \Omega_2/\Omega_e \leq B_1.$$

If V is an algebraic curve Γ (compact Riemann surface) it is easy to see that *equality holds*. For if Γ has genus g, we can find distinct points $P_1, \cdots, P_g \in \Gamma$ such that there exists no nonconstant meromorphic function on Γ with polar divisor at worst $P_1 + \cdots + P_g$; then if $\omega_1, \cdots, \omega_g$ is a basis for the differentials of first kind on Γ and η_i is a normal differential of

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second kind with polar divisor $2P_i$ $(i = 1, \cdots, g)$, no nontrivial linear combination of $\omega_1, \cdots, \omega_g, \eta_1, \cdots, \eta_g$ is exact, so that $\dim \Omega_2/\Omega_e \geq 2g = B_1$.

If V is an arbitrary compact complex manifold, equality need not hold, not even if V is Kähler. For example, let V be a multitorus of dimension n and let $\omega_1, \cdots, \omega_n$ be a basis for the invariant holomorphic differential forms of degree one. Then any meromorphic differential form on V of degree one is of the form $F_1\omega_1 + \cdots + F_n\omega_n$, where each F_i is a meromorphic function on V. If $n > 1$, V may be chosen so that all everywhere meromorphic functions are constant. In this case $\dim \Omega_2/\Omega_e = n$, while $B_1 = 2n$.

However, equality has long been known to hold if V is an algebraic surface, and Kodaira has recently extended this result to algebraic varieties of dimension 3. We shall prove that equality actually holds if V is any compact *Hodge manifold* (i. e. a Kähler manifold in which the basic 2-form is homologous to a scalar multiple of an integral cocycle), which includes all nonsingular algebraic varieties in projective space. The proof we give, a reduction to the known case of algebraic curves by means of the theory of abelian varieties, is quite simple, and possesses the additional virtue of being extendible to algebraic varieties over arbitrary ground fields (in spite of the seemingly crucial role played here by topology), as we intend to show in a later paper.

2. THEOREM. *If V is a compact Hodge manifold, then* $\dim \Omega_2/\Omega_e = B_1$.

Since V is Kähler, B_1 is even, say $B_1 = 2q$, and the vector space of differentials of first kind is of dimension q. Let $\omega_1, \cdots, \omega_q$ be linearly independent differentials of first kind. Define a map Φ of 1-chains on V into points of C^q, the space of q complex variables, by setting

$$\Phi(\gamma) = \left(\int_\gamma \omega_1, \cdots, \int_\gamma \omega_q\right).$$

If H is the first homology group of V with integral coefficients then the periods $\Phi(H)$ are a discrete subgroup of rank $2q$ of the additive group C^q, so $A = C^q/\Phi(H)$ is a multitorus (compact complex Lie group). Since V is a Hodge manifold, A is actually an *abelian variety*, i. e., A is a multitorus that is analytically homeomorphic to an algebraic variety. We define a map ϕ of V into A by fixing a point $p_0 \in V$ and letting, for any $p \in V$, $\phi(p)$ be the image in A of $\Phi(\gamma)$, where γ is any path in V from p_0 to p. ϕ is clearly well defined and is a complex analytic map of V into A. By its definition, ϕ induces an isomorphism of the rational first homology group of V onto that of A. If η is any meromorphic differential on A, then by applying a suitable group translation on A we can get another meromorphic differential

η' that has the property that the polar locus of η' does not contain $\phi(V)$. If η is of the second kind, so is its translate η', and since η and η' have the same periods they differ by an exact differential. For any η of second kind on A whose polar locus does not contain $\phi(V)$, the differential $\phi^{-1}(\eta)$ is a differential of second kind on V. For any cycle γ on V we have

$$\int_\gamma \phi^{-1}(\eta) = \int_{\phi(\gamma)} \eta,$$

so the periods of $\phi^{-1}(\eta)$ are all zero if and only if the periods of η are all zero, i. e., $\phi^{-1}(\eta)$ is exact if and only if η is exact. Thus if $\eta_1, \cdots, \eta_{2q}$ are differentials of second kind on A that are linearly independent modulo exact differentials, then we can assume that $\phi(V)$ is not contained in the polar locus of any η_i, and then $\phi^{-1}(\eta_1), \cdots, \phi^{-1}(\eta_{2q})$ are differentials of second kind on V that are linearly independent modulo exact differentials. That is, our theorem holds for V if it holds for A. We may thus restrict our attention to abelian varieties.

If A is an abelian variety and $\mathfrak{g}$ is a finite subgroup of A, then $A' = A/\mathfrak{g}$ is also an abelian variety. Let π denote the natural homomorphism of A onto A'. If η is a differential of the second kind on A', then $\pi^{-1}(\eta)$ is of the second kind on A. π induces an isomorphism of the rational first homology group of A onto that of A', so the periods of $\pi^{-1}(\eta)$ are all zero if and only if the periods of η are all zero, and if our theorem holds for A' then it also holds for A. Now if A, A' are *arbitrary* abelian varieties such that there exists a finite subgroup $\mathfrak{g}$ of A such that A' is analytically isomorphic to $A/\mathfrak{g}$, then A and A' are said to be *isogenous*. Isogeny is an equivalence relation. Hence if our theorem holds for A then it also holds for any abelian variety isogenous to A.

Let A_1, A_2 be abelian varieties. Then the direct product $A = A_1 \times A_2$ is also an abelian variety. We shall show that the theorem holds for A if and only if it holds for both A_1 and A_2. Bases for the first homology groups of A_1 and A_2 (homeomorphic to the subspaces $A_1 \times (0)$ and $(0) \times A_2$ respectively of A) together give a homology basis for A. Let π_1, π_2 denote the projections of A onto its first and second factors respectively. If η_1, η_2 are differentials of second kind on A_1, A_2 respectively, then $\eta = \pi_1^{-1}(\eta_1) + \pi_2^{-1}(\eta_2)$ is a differential of second kind on A, and the periods of η are all zero if and only if the periods of η_1 and η_2 both are all zero. It follows that if the theorem holds for both A_1 and A_2, then it also holds for A. Conversely, let η be any differential of second kind on A. Then any translate η' of η on A

is also of the second kind and has the same periods as η. If we choose η' such that the polar locus of η' does not contain $(0)\times(0)$, then this polar locus will contain neither $A_1\times(0)$ nor $(0)\times A_2$. η' then induces differentials η_1, η_2 on $A_1\times(0)$, $(0)\times A_2$ respectively, and η_1, η_2 are of the second kind. The differential $\eta'-\pi_1^{-1}(\eta_1)-\pi_2^{-1}(\eta_2)$ has zero period along each cycle in either $A_1\times(0)$ or $(0)\times A_2$, and hence along each cycle in A. Thus for any differential η of second kind on A there exist differentials η_1, η_2 of second kind on A_1, A_2 respectively such that $\eta-\pi_1^{-1}(\eta_1)-\pi_2^{-1}(\eta_2)$ is exact. As a consequence, if the theorem holds for A it must also hold for both A_1 and A_2.

A *simple abelian variety* is one which has no proper abelian subvarieties. Since every abelian variety is isogenous to a direct product of simple abelian varieties, to prove our theorem in complete generality it suffices to prove it for simple abelian varieties. So let A be simple. Let Γ be an algebraic curve in A and let J be the jacobian variety of Γ. Then one knows that there exists an abelian variety B such that J is isogenous to $A\times B$. By the preceding two paragraphs, to prove our theorem for A it suffices to prove it for J.

Let g be the genus of the algebraic curve Γ and let ϕ be the canonical map of Γ into its jacobian variety J. (If we let Γ be the V of the first paragraph of this proof, then $J=A$, and the present ϕ is the same as the preceding one.) Let Γ^g be the direct product $\Gamma\times\Gamma\times\cdots\times\Gamma$ (g times) and let $\Gamma^{(g)}$ be the g-fold symmetric product of Γ. If ω is a differential of second kind on Γ and if $P_1\times\cdots\times P_g$ is a general point of Γ^g, then $\omega(P_1)+\cdots+\omega(P_g)$ is clearly a differential of second kind on Γ^g; since it remains unchanged if we permute the points $P_1,\cdots,P_g$, it is actually a closed meromorphic differential form on the symmetric product $\Gamma^{(g)}$. Since $\Gamma^{(g)}$ and J are birationally equivalent, $\omega(P_1)+\cdots+\omega(P_g)$ defines a closed meromorphic differential form on J. If $q\in J$, we can write $q=\phi(Q_1)+\cdots+\phi(Q_g)$, where $Q_1,\cdots,Q_g\in\Gamma$ and "$+$" refers to the group addition on J; furthermore $Q_1,\cdots,Q_g$ are unique except for order, unless q lies on a certain $(g-2)$-dimensional analytic subvariety W of J. If $q\notin W$ and $Q_1,\cdots,Q_g$ are distinct, then the map $P_1\times\cdots\times P_g\to\phi(P_1)+\cdots+\phi(P_g)$ is an analytic homeomorphism of a neighborhood of $Q_1\times\cdots\times Q_g\in\Gamma^g$ onto a neighborhood of q; thus $\omega(P_1)+\cdots+\omega(P_g)$ is locally exact in a neighborhood of q on J. Next let $q\in J$ be any point not contained in W. Write

$q = \phi(Q_1) + \cdots + \phi(Q_g)$ and let f be a meromorphic function on Γ such that $\omega - df$ is finite at each point $Q_1, \cdots, Q_g$. Then $f(P_1) + \cdots + f(P_g)$ is a meromorphic function on J such that

$$\omega(P_1) + \cdots + \omega(P_g) - d(f(P_1) + \cdots + f(P_g))$$

is a differential on J that is finite in a neighborhood of q, and therefore holomorphic in that neighborhood. As a consequence, $\omega(P_1) + \cdots + \omega(P_g)$ is locally exact at each point of J-W. Finally, let $q \in W$, let U be a small cellular neighborhood of q in J and let $p_0 \in U - U \cap W$ be a point at which $\omega(P_1) + \cdots + \omega(P_g)$ is holomorphic. Then the integral $\int_{p_0}^{p} (\omega(P_1) + \cdots + \omega(P_g))$, taken along a path lying entirely in $U - U \cap W$ is independent of the path (since $U - U \cap W$ is simply connected, W having dimension $g - 2$). Thus we can write $\omega(P_1) + \cdots + \omega(P_g) = dF$ in $U - U \cap W$, where F is meromorphic. By the continuity theorem for meromorphic functions of several complex variables, F is meromorphic in all of U. Therefore $\omega(P_1) + \cdots + \omega(P_g)$ is a differential of the second kind on J. Now note that under the inverse of the map $\phi: \Gamma \to J$, the differential $\omega(P_1) + \cdots + \omega(P_g)$ induces the differential ω on Γ. (If $\phi(\Gamma)$ is contained in the polar locus of $\omega(P_1) + \cdots + \omega(P_g)$ we still get a well defined differential on Γ by applying a constant translation to $\phi(\Gamma)$). Thus if $\omega(P_1) + \cdots + \omega(P_g)$ is exact, then ω is itself exact. It follows that if ω_i $(i = 1, \cdots, 2g)$ are differentials of the second kind on Γ that are linearly independent modulo exact differentials, then the corresponding differentials $\omega_i(P_1) + \cdots + \omega_i(P_g)$ on J are also linearly independent modulo exact differentials. Since the first Betti number of J is $2g$, our theorem holds for J. The proof is complete.

3. It is of interest to note that if ω is a simple differential of second kind on the compact Hodge manifold V, then the singularities of ω are the singularities of exact differentials. More precisely, for each $p \in V$ there exists an everywhere meromorphic function F_p on V such that $\omega - dF_p$ is holomorphic in a neighborhood of p. To prove this, note that it suffices to show that we can find a set of B_1 basic differentials of second kind none of which has a pole at p. The first paragraph of § 2 shows that it suffices to show this if V is an abelian variety. But here we can translate any given set of basic differentials of second kind to get a new such set of differentials none of which has a pole at p.

4. We remark finally that any simple meromorphic differential on a compact Kähler manifold that has exact singularities is of the second kind. More precisely, if V is Kähler and if ω is a meromorphic differential form on V such that for each $p \in V$ there exists a function f_p, meromorphic in a neighborhood of p, such that $\omega - df_p$ is finite in that neighborhood, then ω is automatically closed. For the second degree differential $d\omega$ is of the first kind, so it suffices to prove that all its periods are zero. Let $\sigma = \Sigma_i \alpha_i s_i$ be a singular 2-cycle, where each s_i is a singular simplex and each α_i a constant. To show that $\int_\sigma d\omega = 0$, we may assume that none of the boundaries ∂s_i meets the polar locus of ω and furthermore that each s_i is so small that in a neighborhood U_i of its carrier we have $\omega = \eta_i + df_i$, where η_i is a holomorphic differential in U_i and f_i is a meromorphic function in U_i. Then

$$\int_\sigma d\omega = \Sigma_i \alpha_i \int_{s_i} d\omega = \Sigma_i \alpha_i \int_{s_i} d\eta_i = \Sigma_i \alpha_i \int_{\partial s_i} \eta_i = \Sigma_i \alpha_i \int_{\partial s_i} \omega = \int_{\partial\sigma} \omega = 0,$$

proving our contention. This can be partially generalized to higher degree differentials, but we do not go into this here.

Added in proof. A slightly weaker case of our main result was announced by P. Dolbeault at the second Liège Colloquium on Algebraic Geometry (June 1952).

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NOTE ON FREE MODULAR LATTICES.*

By R. M. THRALL and D. G. DUNCAN.

1. Introduction. The free modular lattice with three generators has 28 elements and its structure is well known.[1] The free modular lattice with four generators has infinite dimension and also has lattice-homomorphic images which have finite dimension but infinitely many elements. It is still an open question just where in passing from the three to the four generator case the boundary between finite and infinite occurs. In this note we work from both directions, but there still remains a gap. On the finite side we study two free modular lattices which are extensions of the free modular lattice with three generators. One of these lattices (type A) is of particular interest as a knowledge of its structure is made use of in the theory of representation of algebras.[2] The generating elements for the two types of lattice under consideration are displayed in the following two figures.

On the infinite side we show in section 5 that the free modular lattice generated by a projective root [3] and an additional element has infinite dimension. This includes as a corollary the special case of three generators and the complement of one.

2. Lattices of Type A. This type of lattice is generated by a finite chain of "diamond" lattices and a single non-comparable element (m). As an aid in displaying the types of elements which occur in lattices of this type we use the notation t_i to stand for any of x_i, y_i, or z_i, whenever this notation is practical. With this notation we find by constructing the tables of unions and meets that these lattices are composed of the following thirteen (essentially distinct) types of elements.

(1) t_i

(2) m

(3) $t_i \cap m$

(4) $t_i \cup m$

* Received December 2, 1952.

[1] G. Birkhoff, *Lattice Theory* (revised edition, 1948), pp. 68-69.

[2] This application occurs in an as yet unpublished investigation into algebras of bounded representation type by R. Brauer and R. M. Thrall.

[3] A projective root is a lattice $P = \{t; x, y, z; w\}$ such that t is the meet and w the join of each pair selected from (x, y, z).

(5) $t_i \cap (m \cup t_j)$ $\quad t_i > t_j$

(6) $x_i \cup (m \cap y_i)$

(7) $x_i \cap (m \cup y_i)$

(8) $(x_i \cap m) \cup (y_i \cap m)$

(9) $(x_i \cap m) \cup (y_i \cap m) \cup t_k$ $\quad x_i > t_k$

(10) $(x_i \cup m) \cap (y_i \cup m)$

(11) $(x_i \cup m) \cap (y_i \cup m) \cap t_k$ $\quad t_k > x_i$

(12) $[(y_i \cup m) \cap x_i] \cup (m \cap y_i)$

(13) $[(x_i \cup m) \cap y_i] \cup (x_j \cap m) \cup (y_j \cap m)$ $\quad i < j$

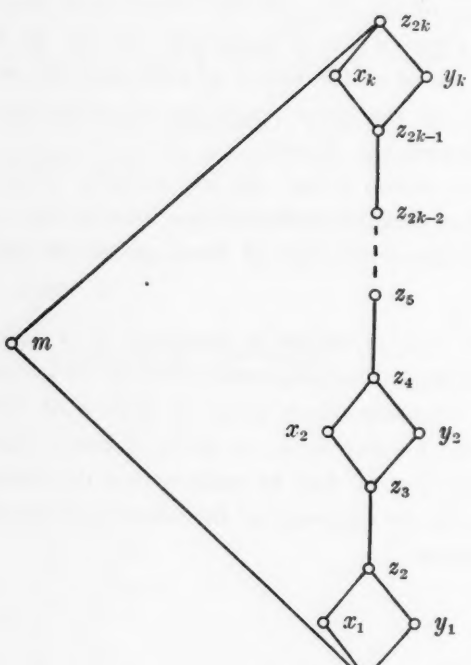

Fig. 1
Type A

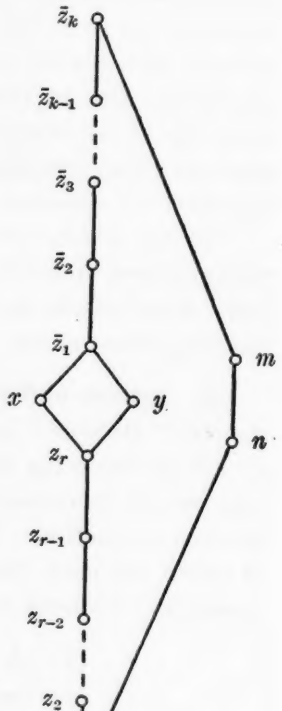

Fig. 2
Type B

Elements (6), (7) and (12) obviously give rise to three essentially analogous elements by interchanges of x_i and y_i. If the generating chain consists of a single diamond we obtain the free modular lattice with three generators.

3. Lattices of Type *B*. For purposes of computation we may think of this type of lattice as being generated by the elements of two lattices of type A, each of which has a single "diamond" embedded in a finite chain and a single non-comparable element; m in one lattice, n in the other. Each generating lattice then consists of twelve types of elements (element 13 of type A is not included since we are dealing here with a single "diamond"). Using the same notation as before we have 23 generating elements (not 24 since the type t_i is common to both generating lattices).

Denoting this set of elements by L, the general lattice of type B is determined by calculating the table of unions ($L \cup L$) and the table of meets ($L \cap L$) and then forming all the unions and meets of these elements with each other until no new elements arise.

The lattice 2+1+1 (Problem 29, G. Birkhoff, *Lattice Theory* (revised edition)) is a lattice of type B with the chains $z_1, \cdots, z_r$ and $\bar{z}_1, \cdots, \bar{z}_k$ both restricted to single elements. This lattice has recently been determined independently by K. Takeuchi who has also constructed the lattice diagram for this lattice of 138 elements (written communication with the authors). Discrepancies occurring between his list of elements and our own are noted in the table of elements given below. This list displays the elements of 2+1+1 with the corresponding type of element for the general type B. Elements obtainable by duality have been omitted (excepting elements 70 and 71). The eleven types of elements in the general type B which have no counterpart in the restricted case 2+1+1 are also given.

It should be noted that not every element of the general type B has a dual unless the length r of the chain below the diamond is equal to the length k of the chain above the diamond.

No.	*Element of* 2+1+1	*Dimension*	*Corresponding element of the general type B*	*Dimension*
1	$z_1 \cap n$	0	$z_i \cap n$	i—1
2	$x \cap n$	1	$x \cap n$	r
3	$z_1 \cap m$	1	$z_i \cap m$	i
4	$y \cap n$	1	$y \cap n$	r
5	$(x \cap n) \cup (z_1 \cap m)$	2	$(x \cap n) \cup (z_i \cap m)$	$r+i$
6	z_1	2	z_i	$3i$—1
7	$(x \cap n) \cup (y \cap n)$	2	$(x \cap n) \cup (y \cap n)$	$r+1$
8	$(y \cap n) \cup (z_1 \cap m)$	2	$(y \cap n) \cup (z_i \cap m)$	$r+i$
9	$(y \cup n) \cap m \cap x$	3	$(y \cup n) \cap m \cap x$	$2r+1$
10	$(x \cap n) \cup z_1$	3	$(x \cap n) \cup z_i$	$2i+r$
11	$n \cap [x \cup (m \cap y)]$	3	$n \cap [x \cup (m \cap y)]$	$r+2$
12	$(x \cap n) \cup (y \cap n) \cup (z_1 \cap m)$	3	$(x \cap n) \cup (y \cap n) \cup (z_i \cap m)$	$r+i+1$

No.	Element of 2+1+1	Dimension	Corresponding element of the general type B	Dimension
13	$(y\cap n)\cup z_1$	3	$(y\cap n)\cup z_i$	$2i+r$
14	$(x\cup n)\cap m\cap y$	3	$(x\cup n)\cap m\cap y$	$2r+1$
15	$x\cap m$	4	$x\cap m$	$2r+2$
16	$x\cap(n\cup y)\cap(m\cup z_1)$	4	$x\cap(n\cup y)\cap(m\cup z_i)$	$2r+i+1$
17	$(y\cup n)\cap[x\cup(n\cap y)]\cap m$	4	$(y\cup n)\cap[x\cup(n\cap y)]\cap m$	$2r+2$
18	$z_2\cap n$	4	$\bar{z}_i\cap n$	$r+i+2$
19	$(z_1\cup n)\cap[(x\cap m)\cup(y\cap m)]$	4	$(z_i\cup n)\cap[(x\cap m)\cup(y\cap m)]$	$r+i+2$
20	$(x\cap n)\cup(y\cap n)\cup z_1$	4	$(x\cap n)\cup(y\cap n)\cup z_i$	$2i+r+1$
21	$(x\cup n)\cap m\cap[y\cup(n\cap x)]$	4	$(x\cup n)\cap[y\cup(n\cap x)]\cap m$	$2r+2$
22	$y\cap(n\cup x)\cap(m\cup z_1)$	4	$y\cap(n\cup x)\cap(m\cup z_i)$	$2r+i+1$
23	$y\cap m$	4	$y\cap m$	$2r+2$
24	$(x\cap m)\cup z_1$	5	$(x\cap m)\cup z_i$	$2r+i+2$
25	$x\cap(n\cup y)$	5	$x\cap(n\cup y)$	$3r+2$
26	$(y\cap n)\cup(x\cap m)$	5	$(y\cap n)\cup(x\cap m)$	$2r+3$
27	$(z_1\cup m)\cap(n\cup y)\cap[x\cup(n\cap y)]$	5	$(z_i\cup m)\cap(n\cup y)\cap[x\cup(n\cap y)]$	$2r+i+2$
28*	$(z_1\cup m)\cap(n\cup x)\cap[y\cup(n\cap x)]$	5	$(z_i\cup m)\cap(n\cup x)\cap[y\cup(n\cap x)]$	$2r+i+2$
29	$(x\cup n)\cap(y\cup n)\cap[(x\cap m)\cup(y\cap m)]$	5	$(x\cup n)\cap(y\cup n)\cap[(x\cap m)\cup(y\cup m)]$	$2r+3$
30	n	5	n	$r+k+3$
31	$(z_2\cap n)\cup(z_1\cap m)$	5	$(\bar{z}_j\cap n)\cup(z_i\cap m)$	$r+i+j+2$
32	$(z_1\cup n)\cap[(m\cap y)\cup x]$	5	$(z_i\cup n)\cap[(m\cap y)\cup x]$	$2i+r+2$
33	$(x\cap n)\cup(y\cap m)$	5	$(x\cap n)\cup(y\cap m)$	$2r+3$
34	$y\cap(n\cup x)$	5	$y\cap(n\cup x)$	$3r+2$
35	$(y\cap m)\cup z_1$	5	$(y\cap m)\cup z_i$	$2r+i+2$
36	$[x\cap(n\cup y)]\cup(x\cap m)$	6	$[x\cap(n\cup y)]\cup(x\cap m)$	$3r+3$
37	$z_1\cup(n\cap y)\cup(m\cap x)$	6	$z_i\cup(n\cap y)\cup(m\cap x)$	$2r+i+3$
38	$(y\cup n)\cap[x\cup(n\cap y)]$	6	$(y\cup n)\cap[x\cup(n\cap y)]$	$3r+3$
39	$(x\cup n)\cap[(x\cap m)\cup(y\cap m)]$	6	$(x\cup n)\cap[(x\cap m)\cup(y\cap m)]$	$2r+4$
40	$(y\cup n)\cap(x\cup n)\cap m\cap z_2$	6	$(y\cup n)\cap(x\cup n)\cap m\cap\bar{z}_i$	$2r+i+3$
41	$(x\cup n)\cap(y\cup n)\cap[(x\cap m)\cup(y\cap m)\cup z_1]$	6	$(x\cup n)\cap(y\cup n)\cap[(x\cap m)\cup(y\cap m)\cup z_i]$	$2r+i+3$
42	$(z_1\cap m)\cup n$	6	$(z_i\cap m)\cup n$	$r+k+i+3$
43	$(z_2\cap n)\cup z_1$	6	$(\bar{z}_j\cap n)\cup z_i$	$r+2i+j+2$
44	$(y\cup n)\cap[(x\cap m)\cup(y\cap m)]$	6	$(y\cup n)\cap[(x\cap m)\cup(y\cap m)]$	$2r+4$
45	$(x\cup n)\cap[y\cup(n\cap x)]$	6	$(x\cup n)\cap[y\cup(n\cap x)]$	$3r+3$
46	$z_1\cup(n\cap x)\cup(m\cap y)$	6	$z_i\cup(n\cap x)\cup(m\cap y)$	$2r+i+3$
47	$[y\cap(n\cup x)]\cup(y\cap m)$	6	$[y\cap(n\cup x)]\cup(y\cap m)$	$3r+3$
48	$x\cap(m\cup y)$	7	$x\cap(m\cup y)$	$3r+4$
49	$(x\cap m)\cup(y\cap n)\cup[x\cap(y\cup n)]$	7	$(x\cap m)\cup(y\cap n)\cup[x\cap(y\cup n)]$	$3r+4$

* This element (No. 28) was missing from Takeuchi's original list and No. 32 of his list was found to be reducible to No. 46 of his list.

No.	*Element of* 2+1+1	*Dimension*	*Corresponding element of the general type B*	*Dimension*
50	$(z_2 \cap n) \cup (x \cap m)$	7	$(\bar{z}_i \cap n) \cup (x \cap m)$	$2r+i+4$
51	$(x \cup n) \cap (y \cup n) \cap [y \cup (m \cap x)]$	7	$(x \cup n) \cap (y \cup n) \cap [y \cup (m \cap x)]$	$3r+4$
52	$(n \cup x) \cap [(m \cap x) \cup (m \cap y) \cup z_1]$	7	$(n \cup x) \cap [(m \cap x) \cup (m \cap y) \cup z_i]$	$2r+i+4$
53	$(x \cap m) \cup (y \cap m)$	7	$(x \cap m) \cup (y \cap m)$	$2r+5$
54	$(y \cup n) \cap (x \cup n) \cap m$	7	$(y \cup n) \cap (x \cup n) \cap m$	$2r+k+4$
55	$(y \cup n) \cap (x \cup n) \cap (m \cup z_1) \cap z_2$	7	$(y \cup n) \cap (x \cup n) \cap (m \cup z_i) \cap \bar{z}_j$	$2r+i+j+3$
56	$z_1 \cup n$	7	$z_i \cup n$	$r+k+2i+3$
57#	$(n \cup y) \cap [(m \cap x) \cup (m \cap y) \cup z_1]$	7	$(n \cup y) \cap [(m \cap x) \cup (m \cap y) \cup z_i]$	$2r+i+4$
58	$(x \cup n) \cap (y \cup n) \cap [x \cup (m \cap y)]$	7	$(x \cup n) \cap (y \cup n) \cap [x \cup (m \cap y)]$	$3r+4$
59	$(z_2 \cap n) \cup (y \cap m)$	7	$(\bar{z}_i \cap n) \cup (y \cap m)$	$2r+i+4$
60	$(y \cap m) \cup (x \cap n) \cup [y \cap (x \cup n)]$	7	$(y \cap m) \cup (x \cap n) \cup [y \cap (x \cup n)]$	$3r+4$
61	$y \cap (m \cup x)$	7	$y \cap (m \cup x)$	$3r+4$
62	x	8	x	$3r+5$
63	$[x \cup (n \cap y)] \cap (m \cup y)$	8	$[x \cup (n \cap y)] \cap (m \cup y)$	$3r+5$
64#	$[(m \cap x) \cup n \cup y] \cap [x \cup \{y \cap m \cap (x \cup n)\}]$	8	$[(m \cap x) \cup n \cup y] \cap [x \cup \{y \cap m \cap (x \cup n)\}]$	$3r+5$
65	$z_2 \cap (n \cup y) \cap (m \cup z_1)$	8	$\bar{z}_j \cap (n \cup y) \cap (m \cup z_i)$	$2r+i+j+4$
66	$(n \cup x) \cap m$	8	$(n \cup x) \cap m$	$2r+k+5$
67	$(y \cup n) \cap [(m \cap y) \cup x]$	8	$(y \cup n) \cap [(m \cap y) \cup x]$	$3r+5$
68	$(z_2 \cap n) \cup (x \cap m) \cup (y \cap m)$	8	$(\bar{z}_i \cap n) \cup (x \cap m) \cup (y \cap m)$	$2r+i+5$
69	$(x \cap m) \cup (y \cap m) \cup z_1$	8	$(x \cap m) \cup (y \cap m) \cup z_i$	$2r+i+5$
70	$(x \cup n) \cap (y \cup n) \cap z_2$ (dual of 69)	8	$(x \cup n) \cap (y \cup n) \cap \bar{z}_i$	$3r+i+4$
71	$(y \cup n) \cap (x \cup n) \cap (m \cup z_1)$ (dual of 68)	8	$(y \cup n) \cap (x \cup n) \cap (m \cup z_i)$	$2r+k+i+4$
72	$(x \cup n) \cap [(m \cap x) \cup y]$	8	$(x \cup n) \cap [(m \cap x) \cup y]$	$3r+5$
73	$(n \cup y) \cap m$	8	$(n \cup y) \cap m$	$2r+k+5$
74	$z_2 \cap (n \cup x) \cap (m \cup z_1)$	8	$\bar{z}_j \cap (n \cup x) \cap (m \cup z_i)$	$2r+i+j+4$
75#	$[(m \cap y) \cup n \cup x] \cap [y \cup \{x \cap m \cap (y \cup n)\}]$	8	$[(m \cap y) \cup n \cup x] \cap [y \cup \{x \cap m \cap (y \cup n)\}]$	$3r+5$
76	$[y \cup (n \cap x)] \cap (m \cup x)$	8	$[y \cup (n \cap x)] \cap (m \cup x)$	$3r+5$
77	y	8	y	$3r+5$

The following eleven types of elements occur in the general lattice of type B but not in the restricted case 2+1+1.

No.	*Element of the general type B.*		*Dimension*
78	$(z_i \cap n) \cup (z_j \cap m)$	$i>j$	$i+j-1$
79	$z_i \cup (z_j \cap n)$	$j>i$	$2i+j-1$
80	$z_i \cup (z_j \cap m)$	$j>i$	$2i+j$
81	$(z_i \cap n) \cup (z_j \cap m) \cup z_s$	$i>j>s$	$i+j+s-1$
82	$(x \cap n) \cup (z_i \cap m) \cup z_j$	$i>j$	$r+i+j$
83	$(y \cap n) \cup (z_i \cap m) \cup z_j$	$i>j$	$r+i+j$

These elements (Nos. 57, 64, 75) were missing from our original list.

No.	*Element of the general type B.*		*Dimension*
84	$(y \cap n) \cup (x \cap n) \cup (z_i \cap m) \cup z_j$	$i>j$	$r+i+j+1$
85	$[x \cup (y \cap m)] \cap (z_i \cup n) \cap (z_j \cup m)$	$i>j$	$r+i+j+2$
86	$[y \cup (x \cap m)] \cap (z_i \cup n) \cap (z_j \cup m)$	$i>j$	$r+i+j+2$
87	$(z_i \cup n) \cap (z_j \cup m) \cap \bar{z}_s$	$i>j$	$r+i+j+s+2$
88	$(z_i \cup n) \cap (z_j \cup m)$	$i>j$	$r+k+i+j+3$

4. The free modular lattice generated by $3+1+1$. This lattice may be considered as being generated by three lattices of the type $2+1+1$ each of which consists of 138 distinct elements. The number of generating elements is then calculated to be 334; this is somewhat less than $3(138)=414$ as there is some duplication. For this number of generating elements the method of direct calculation would not appear to be very fruitful. Apparently the structure of the general case $n+1+1$ will have to be obtained by other methods. Indeed, at present, it is not known for which value n (if any) this lattice becomes infinite. A lattice is said to be of type C if it is generated by a finite diamond lattice and a finite chain. Clearly type C includes types A and B as well as $n+1+1$. It is still an open question as to whether there exists an infinite lattice of type C.

5. An infinite lattice. *Let* $P_1=(t;x,y,z;w)$ and $P_2=(t;x,y,u;w)$ be projective roots. Let L be the free modular lattice generated by P_1 and P_2. We shall show that L has finite dimension by exhibiting lattice-homomorphic images of arbitrarily high dimension.

Let r be any positive integer, let W be the free abelian group with $2r$ generators $\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_r$, let T be the subgroup of W consisting of 0 alone, let X, Y, Z, U be the subgroups generated by $\{\beta_1, \cdots, \beta_r\}$, $\{\alpha_1+\beta_1, \cdots, \alpha_r+\beta_r\}$, $\{\alpha_1, \cdots, \alpha_r\}$, $\{\alpha_1, \alpha_2+\beta_1, \cdots \alpha_r+\beta_{r-1}\}$ respectively. Then clearly $\{T;X,Y,Z;W\}$ and $\{T;X,Y,U;W\}$ are projective roots. Let L_{2r} be the lattice generated by X, Y, Z, U. We shall show that L_{2r} has dimension $2r$.

Let $R_1=Z \cap U$ and let $R_\nu=(((R_{\nu-1} \cup Y) \cap X) \cup U) \cap Z$ $(\nu=2, \cdots, r)$. It is easy to see that R_ν is the free group generated by $\{\alpha_1, \cdots, \alpha_\nu\}$, $(\nu=1, \cdots, r)$. Next let $R_{r+\nu}=((R_\nu \cup Y) \cap X) \cup Z$ $(\nu=1, \cdots, r)$. Then $R_{r+\nu}$ is generated by $\{\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_\nu\}$ $(\nu=1, \cdots, r)$. The chain $T \subset R_1 \subset \cdots \subset R_{2r}=W$ shows that L_{2r} has dimension $2r$, as claimed.

Now L_{2r} is clearly the lattice-homomorphic image of L under the mapping $x \to X$, $y \to Y$, $z \to Z$, $u \to U$, and so L has infinite dimension.

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NILPOTENT CHARACTERISTIC SUBGROUPS OF FINITE GROUPS.*

By REINHOLD BAER.

Every student of the theory of finite groups is familiar with two nilpotent characteristic subgroups: center and hypercenter. Not so well known, though no less important, are Frattini's and Fitting's subgroups. The former, usually denoted by $\phi(G)$, is the intersection of all the maximal subgroups of G; and the latter, denoted by $F(G)$, is the product of all the normal nilpotent subgroups [and is itself nilpotent]. They are quite closely connected, since it may be shown that $F[G/\phi(G)] = F(G)/\phi(G)$ is an elementary abelian group; see Gaschütz [1] or below § 3, Theorem 1.

Hypercenter and Frattini subgroup are distinguished among nilpotent subgroups by a very powerful property: weak hypercentrality. This property may be stated in various equivalent ways [§ 1, Proposition 2] of which we mention the following one: The normal subgroup N of G is weakly hypercentral, if it is nilpotent and if, for every normal subgroup M of G which contains N, the totality of elements x in M, satisfying $x^{[M:N]^i} = 1$ for some i, is a subgroup. Since products of weakly hypercentral normal subgroups need not be weakly hypercentral, it is necessary to survey their totality. It has the following closure property: If M and N are normal subgroups of G such that $M \leq N$, then weak hypercentrality of N in G is necessary and sufficient for weak hypercentrality of M in G and of N/M in G/M [§ 1, Proposition 3]. If we define, as seems sensible, the weak hypercenter of G as the intersection $H_w(G)$ of all the maximal weakly hypercentral normal subgroups of G, then hypercenter, weak hypercenter and Frattini subgroup of $G/H_w(G)$ equal 1 so that $H_w(G)$ contains both the hypercenter and the Frattini subgroup [§ 2, Theorem 1 and § 1, Corollaries 4, 5].

We make use of these results to prove the following theorem which appears to admit of various applications: If M is a minimal normal subgroup of the group G, if $Z(M < G)$ is the centralizer of M in G and if $G/Z(M < G)$ contains a normal subgroup, not 1, whose order is prime to the order of M, then M is abelian and there exists a subgroup S of G such that $G = SZ(M < G)$ and $1 = M \cap S$. The hypothesis concerning the automorphism group $G/Z(M < G)$ of M is indispensable; it is satisfied, for

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instance, whenever $G/Z(M < G)$ contains a soluble normal subgroup different from 1 [§ 4, Proposition 2 and § 5, Lemma 2].

It is natural that these results will prove useful in the study of soluble groups. Consequently we obtain in § 5 a great number of properties characteristic of solubility; and in §§ 6, 7 our results are used to continue the study of n-soluble and n-nilpotent groups which we initiated elsewhere.

Notations. We shall consider finite groups only and the order of the group G [of the group element g] shall be denoted by $o(G)$ [by $o(g)$]. It will be convenient to call an element x an *n-element,* if $x^{n^i} = 1$ for some integer i; in other words: if every prime divisor of $o(x)$ is a factor of n. Similarly we shall say that x is a *Pn-element,* if $o(x)$ is prime to n. A group will be termed an *n-group* [a *Pn-group*], if all its elements are n-elements [Pn-elements].

$$Z(G) = \text{center of } G.$$

$$Z(A < G) = \text{centralizer of } A \text{ in } G.$$

$$H(G) = \text{hypercenter of } G.$$

The hypercenter of G may be defined in various equivalent ways. It is, for instance, the intersection of all the normal subgroups N of G which satisfy $Z(G/N) = 1$; and it is also the terminal member of the ascending center chain. Subsets of the hypercenter shall be referred to as hypercentral subsets. For the elementary properties of these concepts see, for instance, Baer [5; section 5].

$$[G, G] = \text{commutator subgroup of } G.$$

The subgroup S of G is a *complement* of the normal subgroup N of G, if $G = NS$ and $1 = N \cap S$.

1. The weakly hypercentral subgroups. We begin by recalling some important concepts. If the totality of n-elements [Pn-elements] in the group G forms a subgroup of G, then we term this characteristic subgroup of G the *n-component* G_n [the *Pn-component* G'_n] of G. If the group G is in particular nilpotent, then these components exist for every n and G is their direct product, see, for instance, Zassenhaus [1; p. 107, Satz 11]. The subgroup S of G is termed a *Pn-complement of* G, if S is an n-group and $[G:S]$ is prime to n. One verifies easily that a subgroup T of G is the n-component of G if, and only if, T is a normal subgroup and a Pn-complement of G.

Like the components these complements need not exist; but whereas components are unique, there may exist several Pn-complements.

LEMMA 1. *Assume that the nilpotent normal subgroup M of G is part of the normal subgroup N of G and that j is a multiple of $[N:M]$. Then*

(a) *there exists a Pj-complement of N and*

(b) *$G = TM'_j$ for every normalizer T of a Pj-complement of N.*

Proof. M'_j is a normal subgroup of G, since it is a characteristic subgroup of the normal subgroup M of G. Since N/M and $M/M'_j \simeq M_j$ are j-groups whereas M'_j is a Pj-group, $o(M'_j)$ and $[N:M'_j]$ are relatively prime. Consequently we may apply Schur's Theorem asserting the existence of a subgroup S of N such that $N = SM'_j$ and $1 = S \cap M'_j$; see, for instance, Zassenhaus [1; p. 125, Satz 25]. Then $S \simeq N/M'_j$ and one sees easily that S is a Pj-complement of N.

If U is any Pj-complement of N, then $o(U) = [N:M'_j]$ and this implies that U too is a complement of M'_j in N since $o(N) = o(U)o(M'_j)$ and $(o(U), o(M'_j)) = 1$. Denote by T the normalizer of U in G. If g is an element in G, then g transforms the normal subgroups N and M'_j into themselves. Hence

$$N = g^{-1}UM'_jg = [g^{-1}Ug]M'_j, \qquad 1 = g^{-1}(U \cap M'_j)g = [g^{-1}Ug] \cap M'_j.$$

Thus $g^{-1}Ug$ is also a complement of M'_j in N. Since M'_j is soluble as a subgroup of the nilpotent group M, we may apply the Theorem of Witt-Zassenhaus; see Zassenhaus [1; p. 126, Satz 27]. Hence U and $g^{-1}Ug$ are conjugate in N; and there exists therefore an element w in N such that $w^{-1}Uw = g^{-1}Ug$. From $N = UM'_j$ we deduce the existence of elements u and t in U and M'_j respectively such that $w = ut$. Consequently

$$g^{-1}Ug = w^{-1}Uw = t^{-1}u^{-1}Uut = t^{-1}Ut$$

so that gt^{-1} belongs to the normalizer T of U. The element g belongs therefore to $Tt \leq TM'_j$. Hence $G = TM'_j$; and this completes the proof.

DEFINITION. *The normal subgroup N of G is weakly hypercentral, if it has the following property:*

(W) *If N is part of the normal subgroup M of G, if x and y are elements in N and M respectively, and if $o(x)$ is prime to $[M:N]$ and to $o(y)$, then $xy = yx$.*

A justification of the term "weakly hypercentral" may be found in the following simple and important result.

PROPOSITION 1. *Hypercentral normal subgroups are weakly hypercentral and weakly hypercentral normal subgroups are nilpotent.*

Proof. The normal subgroup N of G is hypercentral if, and only if, every element in N commutes with every element of relatively prime order in G; see, for instance, Baer [5, § 5, Theorem 3]. It is clear that this condition implies (W) and that therefore hypercentrality implies weak hypercentrality. If the normal subgroup N of G is weakly hypercentral and if we let $M = N$ in (W), then it follows that elements of relatively prime order in N commute; and it is well known that this condition is necessary and sufficient for nilpotency of N; see, for instance, Baer [1] or Zassenhaus [1].

PROPOSITION 2. *The following properties of the normal subgroup N of G are equivalent.*

(i) *N is weakly hypercentral.*

(ii) *N is nilpotent; and if N is part of the normal subgroup M of G, then the totality of $[M:N]$-elements in M is a subgroup of M.*

(iii) *If N is part of the normal subgroup M of G, and if j is a multiple of $[M:N]$, then M is the direct product of its j-component M_j and its Pj-component M'_j—note that $M'_j \leq N$.*

Proof. Assume first that N is weakly hypercentral. Then N is nilpotent by Proposition 1. Suppose now that N is part of the normal subgroup M of G and that $n = [M:N]$. We deduce from Lemma 1 the existence of a Pn-complement S of M; and if T is the normalizer of S in G, then $G = TN'_n$. Since every element in N'_n is a Pn-element and every element in S is an n-element, we deduce from the weak hypercentrality of N that every element in N'_n commutes with every element in S. Hence $N'_n \leq T$ and we deduce $G = T$ from $G = TN'_n$. Consequently S is a normal subgroup of G and hence of M; and this implies that the Pn-complement S of M is the n-component of M. Thus we have shown that (ii) is a consequence of (i).

Assume next the validity of (ii). Suppose that the normal subgroup M of G contains N and that j is a multiple of $[M:N]$. Since N is nilpotent, N is the direct product of its j-component N_j and its Pj-component N'_j. These are characteristic subgroups of the normal subgroup N of G; and as such they are normal subgroups of G. Next it follows from (ii) that the

totality V of $[M:N]$-elements in M is a characteristic subgroup of M. Since the normal subgroups N_j and V of M are both j-groups—here we use the fact that j is a multiple of $[M:N]$—their product VN_j is a normal j-subgroup of M. Since V contains every $[M:N]$-element in M, we have $M = NV = N'_j(N_jV)$. Hence M is the product of the normal Pj-group N'_j and the normal j-group N_jV; and now it is clear that M is the direct product of a Pj-group and a j-group. But these are necessarily components of M; and thus we see that (iii) is a consequence of (ii).

That finally (i) is a consequence of (iii), is almost obvious. This completes the proof.

COROLLARY 1. *Suppose that the weakly hypercentral normal subgroup M of G is part of the normal subgroup N of G. Then the k-elements in N form a subgroup of N if, and only if, the k-elements of N/M form a subgroup of N/M.*

Proof. The totality of k-elements in N is a subgroup of N if, and only if, the product of any two k-elements in N is a k-element in N. But this property is clearly invariant under homomorphisms, since k-elements in N/M may be represented by k-elements in N. The totality of k-elements in N/M is therefore certainly a subgroup of N/M whenever the totality of k-elements in N is a subgroup of N.

Assume conversely that the totality of k-elements in N/M is a subgroup K/M of N/M. It is clear that K/M is a characteristic subgroup of the normal subgroup N/M of G/M. Hence K/M is a normal subgroup of G/M and K is a normal subgroup of G. Since K/M is a k-group, $[K:M]$ is a divisor of a suitable power j of k. Since M is weakly hypercentral, we may apply Proposition 2, (iii). Hence K is the direct product of a j-group J and a Pj-group J'. Since J is the j-component of K, J is also the k-component of K. Since every k-element in N represents a k-element in N/M, and since the latter elements belong to K/M, it follows that K contains every k-element in N. Hence J is the k-component of N; and we have shown that the totality of k-elements in N is a subgroup of N, namely J. This completes the proof.

COROLLARY 2. *Supose that the weakly hypercentral normal subgroup M of G is part of the normal subgroup N of G. Then N is nilpotent if, and only if, N/M is nilpotent.*

Proof. A group H is nilpotent if, and only if, the p-elements in H

form, for every prime p, a subgroup of H; see, for instance, Zassenhaus [1]. But it follows from Corollary 1 and the weak hypercentrality of M that the p-elements in N form a subgroup if, and only if, the p-elements in N/M form a subgroup. Hence nilpotency of N and of N/M are equivalent properties.

We note that Corollary 2 generalizes a result due to Gaschütz [1; Satz 10].

COROLLARY 3. *If N is a normal subgroup of G, and if $NZ(N<G)$ is a weakly hypercentral normal subgroup of G, then $G/NZ(N<G)$ does not contain normal subgroups different from 1 whose order is prime to $o(N)$.*

Proof. Assume that the normal subgroup M of G contains $NZ(N<G)$ and that $n=[M:NZ(N<G)]$ is prime to $o(N)$. It follows from the weak hypercentrality of $NZ(N<G)$ and Proposition 2 that M is the direct product of an n-group M_n and a Pn-group M'_n. Since M'_n is the Pn-component of M, and since N is a Pn-subgroup of M, we have $N\leq M'_n\leq NZ(N<G)$. From the first inequality we infer that M_n is part of $Z(N<G)$. Hence $M\leq NZ(N<G)$ implying $M=NZ(N<G)$ and $n=1$, as we wanted to show.

Remark. If in particular N is a normal p-subgroup of G such that $NZ(N<G)$ is weakly hypercentral, then it follows from Corollary 3 that the order of every normal subgroup, not 1, of $G/NZ(N<G)$ is divisible by p. If $M/NZ(N<G)$ is a nilpotent normal subgroup of $G/NZ(N<G)$, then it is the direct product of a p-group and a Pp-group both of which are normal. It follows that the latter is 1, proving that nilpotent normal subgroups of $G/NZ(N<G)$ are always p-groups.

PROPOSITION 3. *If M and N are normal subgroups of G and $M\leq N$, then weak hypercentrality of N [in G] is necessary and sufficient for weak hypercentrality of M [in G] and of N/M [in G/M].*

Proof. Assume first that N is a weakly hypercentral normal subgroup of G. Consider a normal subgroup K of G which contains M and elements x and y in M and K respectively such that $o(x)$ is prime to $[K:M]$ and $o(y)$. From $KN/N\simeq K/(N\cap K)\cong[K/M]/[(N\cap K)/M]$ we deduce that $[KN:N]$ is a factor of $[K:M]$. Hence $o(x)$ is also prime to $[KN:N]$. But the normal subgroup KN of G contains the weakly hypercentral normal subgroup N of G; x belongs to M and hence to N and y belongs to K and hence to KN. Application of condition (W) shows $xy=yx$; and this proves weak hypercentrality of M in G. Consider next a normal subgroup H/M of G/M which

contains N/M and consider elements U and V in N/M and H/M respectively such that $o(U)$ is prime to $[H/M : N/M]$ and $o(V)$. There exist elements u and v such that $U = Mu$, $V = Mv$ and such that u is an $o(U)$-element, v is an $o(V)$-element. Since $[H/M : N/M] = [H : N]$, $o(u)$ is prime to $[H : N]$ and $o(v)$; and we may deduce $uv = vu$ from the weak hypercentrality of N in G. But then $UV = VU$; and this proves the weak hypercentrality of N/M in G/M.

Assume conversely that M is a weakly hypercentral subgroup of G and that N/M is a weakly hypercentral subgroup of G/M. Consider a normal subgroup K of G which contains N and let $k = [K : N]$. Since the normal subgroup K/M of G/M contains the weakly hypercentral normal subgroup N/M of G/M, and since $[K/M : N/M] = [K : N] = k$, we deduce from Proposition 2, (ii) that the totality of k-elements in K/M is a characteristic subgroup H/M of K/M and hence a normal subgroup of G/M. Thus H is a normal subgroup of G which contains the weakly hypercentral normal subgroup M of G; and it follows from Proposition 2, (ii) that the totality of $[H : M]$-elements in H is a subgroup T of H. Since H/M is a k-group, $[H : M]$-elements are k-elements. Hence T is a k-group. If furthermore z is a k-element in K, then Mz is a k-element in K/M and belongs therefore to H/M. Hence z is a k-element in H. To show that z is also an $[H : M]$-element, and therefore in T, we prove that k is a factor of $[H : M]$, as follows. From our choices of H and T it follows [by Proposition 2, (iii)] that $K/M = (H/M)(N/M)$ or $K = HN$ and $H = TM$. Hence $K = TN$ and consequently $k = [K : N] = [T : (T \cap N)]$ is a factor of $[T : (T \cap M)] = [H : M]$, as we wanted to show. This completes the proof of the fact that T is the totality of k-elements in K; and thus we have verified the validity of Proposition 2, (ii). Hence N is weakly hypercentral, as we wanted to show. This completes the proof.

PROPOSITION 4. *The normal subgroup N of G is weakly hypercentral if, and only if, N is a nilpotent group whose primary components are weakly hypercentral.*

The necessity of our conditions is an immediate consequence of Propositions 1 and 3 and their sufficiency will be an immediate consequence of the following fact.

LEMMA 2. *If M and N are weakly hypercentral normal subgroups of G, and if $o(M)$ and $o(N)$ are relatively prime, then MN is weakly hypercentral.*

Proof. Consider a normal subgroup Q of G which contains MN and suppose that x and y are elements in MN and Q respectively such that $o(x)$ is prime to $[Q:MN]$ and to $o(y)$. Since x is in MN, and since $o(M)$ and $o(N)$ are relatively prime, there exist uniquely determined elements m and n in M and N respectively such that $x = mn$. We note that MN is the direct product of M and N and that $o(x) = o(m)o(n)$. From

$$[Q:M] = [Q:MN][MN:M] = [Q:MN]o(N)$$

we infer that $o(m)$ is prime to $[Q:M]$, since $o(M)$ is prime to $o(N)$ and since $o(m)$ as a divisor of $o(x)$ is prime to $[Q:MN]$. Since $o(x)$ is prime to $o(y)$, $o(m)$ is prime to $o(y)$ too; and now we infer from the weak hypercentrality of M that $my = ym$. Similarly we see that $ny = yn$. Hence $xy = yx$; and this proves that MN is weakly hypercentral. This completes the proof of Lemma 2 and, as we mentioned before, of Proposition 4.

Example 1. Denote by p an odd prime and by D the direct product of two cyclic groups of order p. The group G arises from D by adjunction of an element s satisfying

$$s^2 = 1,\ s^{-1}ds = d^{-1} \text{ for every } d \text{ in } D.$$

One verifies easily that every cyclic subgroup of D is a weakly hypercentral subgroup of G whereas D itself is not weakly hypercentral. This example shows that Lemma 2 ceases to be true if we omit from it the hypothesis that $o(M)$ and $o(N)$ are relatively prime.

Example 1 shows furthermore that there may exist several maximal weakly hypercentral normal subgroups and that the product of all weakly hypercentral normal subgroups need not be weakly hypercentral. Thus we are led to the following definition: *The weak hypercenter $H_w(G)$ of G* is the intersection of all the maximal weakly hypercentral normal subgroups of G.

PROPOSITION 5. *The following two properties of the normal subgroup N of G are equivalena.*

(i) $N \leq H_w(G)$.

(ii) *If M is a weakly hypercentral normal subgroup of G, then MN is weakly hypercentral.*

Proof. Assume first that $N \leq H_w(G)$ and that M is a weakly hypercentral normal subgroup of G. There exists a maximal weakly hypercentral

normal subgroup R of G which contains M. Then $N \leq H_w(G) \leq R$ by the definition of $H_w(G)$. The normal subgroup MN is therefore part of the weakly hypercentral normal subgroup R of G; and MN is weakly hypercentral by Proposition 3. Thus (ii) is a consequence of (i).

Assume conversely that N meets requirement (ii) and that U is a maximal weakly hypercentral normal subgroup of G. We deduce from (ii) that NU is a weakly hypercentral normal subgroup of G; and this implies $U = NU$ or $N \leq U$ because of the maximality of U. Hence N is part of the intersection $H_w(G)$ of all the maximal weakly hypercentral normal subgroups of G. Thus (i) is a consequence of (ii).

Remark. It might be worth pointing out that the preceding argument is purely lattice theoretical and is applicable in the following situation: a lattice $\wedge$ and subset $\mathfrak{S}$ of $\wedge$ which has the following two properties.

(a) If σ is in $\mathfrak{S}$, then every part of σ belongs to $\mathfrak{S}$.

(b) Every element in $\mathfrak{S}$ is contained in some maximal element in $\mathfrak{S}$.

COROLLARY 4. $H(G) \leq H_w(G)$.

Proof. If N is a weakly hypercentral normal subgroup of G, then $NH(G)/N \leq H(G/N)$ and it follows from Propositions 1 and 3 that $NH(G)/N$ is a weakly hypercentral normal subgroup of G/N. Now we deduce from the weak hypercentrality of N and from Proposition 3 that $NH(G)$ is weakly hypercentral. Thus condition (ii) of Proposition 5 is satisfied by $H(G)$, proving $H(G) \leq H_w(G)$.

PROPOSITION 6. *If the normal subgroup N of G is part of $H_w(G)$, then* $H_w(G/N) = H_w(G)/N$.

Proof. One deduces easily from Proposition 3 that M/N is a maximal weakly hypercentral normal subgroup of G/N if, and only if, M is a maximal weakly hypercentral normal subgroup of G; and from this fact $H_w(G/N) = H_w(G)/N$ is an easy consequence since N is part of every maximal weakly hypercentral normal subgroup of G.

COROLLARY 5. $H_w[G/H_w(G)] = H[G/H_w(G)] = 1$.

This is a simple consequence of Corollary 4 and Proposition 6.

2. Frattini's subgroup. Frattini's subgroup is *the intersection $\phi(G)$ of all the maximal subgroups of the group G.* Since finite groups, not 1,

possess maximal subgroups, $1 < G$ implies $\phi(G) < G$; and it is well known that this characteristic subgroup $\phi(G)$ is always nilpotent; see, for instance, Zassenhaus [1; p. 115].

PROPOSITION 1. *The following two properties of the subset S of the group G are equivalent.*

(i) $S \leq \phi(G)$.

(ii) *If T is a subgroup of G and $G = \{S, T\}$, then $T = G$.*

The simple proof of this well known property may be left to the reader; see Zassenhaus [1; p. 45].

For a further analysis we need some criteria concerned with the complements of minimal normal subgroups. We recall that the subgroup S of G is termed a *complement* of the normal subgroup N of G, if $G = NS$ and $1 = N \cap S$.

LEMMA 1. *The following properties of the abelian minimal normal subgroup M of G and the subgroup S of G are equivalent.*

(i) *S is a complement of M in G.*

(ii) *S is a maximal subgroup of G and $M \not\leq S$.*

(iii) *$G = MS$ and $S \neq G$.*

(iv) *$G = MS$ and $M \not\leq S$.*

Proof. We assume first that $G = MS$ and $1 = M \cap S$. Then $M \neq 1$ is certainly not part of S. Consider now a subgroup T of G such that $S < T$. Then there exists an element t in T which does not belong to S. From $G = MS$ we deduce the existence of elements r and s in M and S respectively such that $t = rs$. From $S < T$ it follows that r is in $M \cap T$; and $r \neq 1$, since otherwise t would be in S. Since M is a minimal normal subgroup of G, M is generated by the elements conjugate to r in G. Since M is abelian and $G = MS$, every element conjugate to r in G is obtained by transforming r by elements in S. Consequently $M \leq \{S, r\} \leq T$. Hence $G = MS = T$ proving that S is a maximal subgroup of G. Thus (ii) is a consequence of (i).

It is clear that (ii) implies (iii) and that (iii) implies (iv).

Assume finally the validity of (iv) and let $J = M \cap S$. Then every element in M transforms J into itself, since M is abelian; and every element in S transforms J into itself, since M is a normal subgroup of G and since

therefore J is a normal subgroup of S. From $G = MS$ we deduce now that J is a normal subgroup of G. Since M is not part of S, $J < M$; and now we deduce $J = 1$ from the minimality of M. Hence S is a complement of M in G; and this completes the proof.

Remark 1. Without the hypothesis, that M be abelian, Lemma 1 ceases to be true as may be seen from easily constructed examples.

Remark 2. If the minimal normal subgroup M of G is contained in every maximal subgroup of G [does not satisfy the preceding condition (ii) for any S], then $M \leq \phi(G)$. Since $\phi(G)$ is nilpotent and $M \neq 1$, it follows that $M \cap Z[\phi(G)] \neq 1$. But $Z[\phi(G)]$ is a characteristic subgroup of a characteristic subgroup of G. Hence $Z[\phi(G)]$ is a characteristic subgroup and $M \cap Z[\phi(G)]$ is a normal subgroup of G. Now it follows from the minimality of M that $M \cap Z[\phi(G)] = M$ or $M \leq Z[\phi(G)]$; and this implies in particular the commutativity of M.

Remark 3. If the minimal normal subgroup M of G is abelian, then M is an abelian group without proper characteristic subgroups. If the prime number p divides the order of M, then M^p is a characteristic subgroup of M and $M^p < M$. Hence $M^p = 1$. It follows that the product of all abelian minimal normal subgroups of G is an abelian group the orders of whose elements are squarefree; and we recall that such an abelian group is called *elementary*.

PROPOSITION 2. *The normal subgroup N of G is the Frattini subgroup of G if, and only if, N has the following properties:*

(a) *If S is a subgroup of G and $G = NS$, then $S = G$.*

(b) *Every abelian minimal normal subgroup of G/N possesses a complement in G/N.*

Remark 4. This result is closely related to Gaschütz [1; Satz 14].

Proof. That $\phi(G)$ has property (a), is a consequence of Proposition 1. Let $G^* = G/\phi(G)$. Then it is clear that $\phi(G^*) = 1$. If M is an abelian minimal normal subgroup of G^*, then we infer from $\phi(G^*) = 1$ and $M \neq 1$ the existence of a maximal subgroup T of G^* which does not contain M. It follows from Lemma 1 that T is a complement of M in G^*. Thus $\phi(G)$ has properties (a) and (b).

Assume conversely that the normal subgroup N of G has properties (a)

and (b). Then we deduce $N \leq \phi(G)$ from (a) and Proposition 1. We let next $G^* = G/N$ and assume by way of contradiction that $\phi(G^*) \neq 1$. Since $\phi(G^*)$ is nilpotent, its center $Z[\phi(G^*)]$ is not 1 either. But the center of the Frattini subgroup is a characteristic subgroup. Consequently there exists a minimal normal subgroup M of G^* which is part of $Z[\phi(G^*)]$. Clearly M is abelian and we deduce from (b) the existence of a complement S of M in G^*. From $MS = G^*$ and $M \leq \phi(G^*)$ we deduce $S = G^*$ [by Proposition 1]. But then $1 = M \cap S = M \cap G^* = M$, an impossibility. Hence $\phi(G^*) = 1$. Consequently N is the intersection of all those maximal subgroups of G which contain N and this implies clearly $\phi(G) \leq N$. This completes the proof of $N = \phi(G)$.

THEOREM 1. $\phi(G) \leq H_w(G)$.

Proof. We recall that $\phi(G)$ is nilpotent. Hence $\phi(G)$ is the direct product of its primary components ϕ_p. We prove first the following fact.

(1) If P is a weakly hypercentral normal p-subgroup of G, then $P\phi_p$ is weakly hypercentral.

Since P and ϕ_p are normal p-subgroups of G, $P\phi_p$ is likewise a normal p-subgroup of G. We want to verify the validity of condition (ii) of § 1, Proposition 2. Consider therefore a normal subgroup N of G which contains $P\phi_p$. If $[N : P\phi_p]$ is not prime to p, then every element in N is an $[N : P\phi_p]$-element and there is nothing to prove. We assume therefore that $k = [N : P\phi_p]$ is prime to p. Since p-groups are nilpotent, we may apply § 1, Lemma 1. Consequently there exists a Pk-complement K of N; and if H is the normalizer of K in G, then $G = HP\phi_p$. Since ϕ_p is part of $\phi(G)$, we may apply Proposition 1. Hence $G = HP$.

Now we let $KP = U$. If x is an element in P, then $U = x^{-1}Ux$, since P is part of U. If y is an element in H, then $P = y^{-1}Py$, since P is a normal subgroup of G; and $K = y^{-1}Ky$, since H is the normalizer of K. Hence $U = y^{-1}Uy$ for y in H. From $G = HP$ it follows now that every element in G transforms U into itself; in other words: $U = KP$ is a normal subgroup of G.

Since K is a Pk-complement of N, K is a k-group whose index $[N : K]$ is prime to k. We recall next that $k = [N : P\phi_p]$ is prime to p and that $P\phi_p$ is a p-group. Consequently K is a complement of $P\phi_p$ in N and in particular K is a Pp-group.

From $U/P \simeq K/(K \cap P) = K$ we deduce that $[U : P]$ is prime to p.

Next we recall that P is weakly hypercentral. It follows therefore from § 1, Proposition 2, (ii) that the totality of $[U:P]$-elements in U is a subgroup V of U. But P is a p-group and $[U:P]$ is prime to p. Hence V is the totality of Pp-elements in U. Since the Pp-group K is a complement of P in U, one verifies that $K = V$ is the totality of Pp-elements in U. Hence K is a characteristic subgroup of the normal subgroup U of G; and this proves that K is a normal subgroup of G.

We recall now that the normal subgroup K of G is a complement of $P\phi_p$ in N. Thus N is the direct product of the p-group $P\phi_p$ and the Pp-group K. It is clear now that K is the totality of $[N:P\phi_p]$-elements in N. Thus we have verified the validity of condition (ii) of § 1, Proposition 2; and this completes the proof of (1).

(2) If N is a weakly hypercentral normal subgroup of G, then $N\phi(G)$ is weakly hypercentral.

From the weak hypercentrality of N we deduce the nilpotency of N [§ 1, Proposition 1]. Thus N is the direct product of its primary components N_p; and it follows from § 1, Proposition 2 that every N_p is weakly hypercentral. We deduce from (1) that $N_p\phi_p$ is weakly hypercentral. It is clear that $N\phi(G)$ is a nilpotent normal subgroup of G with p-component $N_p\phi_p$. Application of § 1, Proposition 4 proves now the weak hypercentrality of $N\phi(G)$.

(2) is equivalent to § 1, Proposition 5, (ii). Thus $\phi(G) \leq H_w(G)$ is a consequence of § 1, Proposition 5.

COROLLARY 1. *The Frattini subgroup is weakly hypercentral.*

This is an immediate consequence of Theorem 1 and § 1, Proposition 3.

COROLLARY 2. *G is an n-group if, and only if, $G/\phi(G)$ is an n-group.*

Proof. It is clear that $G/\phi(G)$ is an n-group whenever G is an n-group. Assume conversely that $G/\phi(G)$ is an n-group. By Corollary 1, $\phi(G)$ is weakly hypercentral; and hence it follows from § 1, Proposition 2 that G is the direct product of a $[G:\phi(G)]$-group N and a $P[G:\phi(G)]$-group N'. It is clear that $N' \leq \phi(G)$ and $G = NN'$; and now it follows from Proposition 1 that $G = N$ is a $[G:\phi(G)]$-group. Since $G/\phi(G)$ is an n-group, G itself is an n-group, as we wanted to show.

COROLLARY 3. *Suppose that m and n are relatively prime integers.*

Then G is the direct product of an m-group and an n-group if, and only if, $G/\phi(G)$ is the direct product of an m-group and an n-group.

Proof. We note first that the group H is the direct product of an m-group and an n-group if, and only if, H satisfies the following three conditions:

(a) H is an mn-group;

(b) the m-elements in H form a subgroup;

(c) the n-elements in H form a subgroup.

By Corollary 2, (a) is satisfied by G if, and only if, (a) is satisfied by $G/\phi(G)$. By Corollary 1, $\phi(G)$ is weakly hypercentral; and consequently we deduce from § 1, Corollary 1 that (b) and (c) are satisfied by G if, and only if, (b) and (c) are satisfied by $G/\phi(G)$. Of these various equivalences Corollary 3 is an immediate consequence.

3. Fitting's subgroup. Following Fitting and Wendt we denote by $F(G)$ *the product of all the nilpotent normal subgroups of* G.

PROPOSITION 1. *$F(G)$ is nilpotent.*

Proof. Since the Sylow subgroups of nilpotent subgroups are themselves nilpotent normal subgroups, and since nilpotent groups are direct products of their Sylow subgroups, it is clear that $F(G)$ is the product of all the normal subgroups of prime power order. But the product of normal p-subgroups is itself a normal p-subgroup. $F(G)$ is therefore the direct product of p-groups [for various primes p] and as such $F(G)$ is nilpotent.

For another proof of this fact see Fitting [1; p. 102, Satz 14].

PROPOSITION 2. *$Z[F(G) < G]/Z[F(G)]$ does not contain soluble normal subgroups different from* 1.

Proof. We note first that

$$Z[F(G)] = F(G) \cap Z[F(G) < G] \leq Z(Z[F(G) < G]).$$

Assume by way of contradiction that $W = Z[F(G) < G]/Z[F(G)]$ contains soluble normal subgroups different from 1. Then we form the product of all the soluble normal subgroups of W which is itself a soluble characteristic subgroup C of W. From $C \neq 1$ and the solubility of C we deduce the existence of an abelian characteristic subgroup $A \neq 1$ of C. Then

$A = B/Z[F(G)]$ where $Z[F(G)] < B \leq Z[F(G) < G]$ and where B is a characteristic subgroup of G. Furthermore $Z[F(G)] \leq Z(B)$ and $B/Z[F(G)]$ is abelian. Hence B is nilpotent so that $B \leq F(G) \cap Z[F(G) < G] = Z[F(G)]$, a contradiction.—For another proof of this fact see Fitting [1; p. 105, Hilfssatz 12].

THEOREM 1. *$F[G/\phi(G)] = F(G)/\phi(G)$ is an elementary abelian group, so that $\phi[F(G)] \leq \phi(G)$.*

Proof. Let $F[G/\phi(G)] = W/\phi(G)$. It is clear that $\phi(G)$ is part of $F(G)$ and that $F(G)/\phi(G)$ is nilpotent. Hence $F(G) \leq W$. We deduce from § 2, Corollary 1 that $\phi(G)$ is weakly hypercentral; and we deduce now from § 1, Corollary 2 that the nilpotency of W is a consequence of the nilpotency of $W/\phi(G) = F[G/\phi(G)]$. Thus $W \leq F(G)$ completing the proof of $W = F(G)$ or of $F[G/\phi(G)] = F(G)/\phi(G)$.

Now we let $G^* = G/\phi(G)$. Then $\phi(G^*) = 1$. Assume by way of contradiction that $F(G^*)$ is not abelian. Then the commutator subgroup $C = [F(G^*), F(G^*)]$ of $F(G^*)$ is different from 1. The intersection D of C and the center $Z[F(G^*)]$ of the nilpotent group $F(G^*)$ is therefore likewise different from 1. Since C and $Z[F(G^*)]$ are characteristic subgroups of $F(G^*)$, D is likewise a characteristic subgroup of the characteristic subgroup $F(G^*)$. Hence D is a characteristic subgroup, not 1, of G^*. Consequently D contains a minimal normal subgroup M of G^*. It is clear that M is abelian, since M is part of the center of $F(G^*)$. From $\phi(G^*) = 1$ we infer the existence of a maximal subgroup S of G^* which does not contain M; and it follows from § 2, Lemma 1 that S is a complement of M in G^*. From $M \leq F(G^*)$ and Dedekind's Law we infer now that

$$F(G^*) = G^* \cap F(G^*) = MS \cap F(G^*) = M[S \cap F(G^*)].$$

Since M is contained in the center of $F(G^*)$, every element in M commutes with every element in $S \cap F(G^*)$; and it follows now that $F(G^*)$ is the direct product of the abelian group M and of $S \cap F(G^*)$. The commutator subgroup C of $F(G^*)$ is consequently contained in $S \cap F(G^*)$. We recall now that $M \leq D \leq C$ and that S is a complement of M in G^*. Hence $M = M \cap C \leq M \cap S = 1$, a contradiction which proves the commutativity of $F(G^*)$.

As an abelian group $F(G^*)$ is the direct product of its primary components. If P is the p-component of $F(G^*)$, then P^p is a characteristic subgroup of the characteristic subgroup P of the characteristic subgroup $F(G^*)$ of G^*. Hence P^p is a characteristic subgroup of G^*. Assume now by

way of contradiction that $P^p \neq 1$. Then P^p contains a minimal normal subgroup Q of G^* which is necessarily abelian. From $\phi(G^*) = 1$ we deduce the existence of a maximal subgroup R of G^* which does not contain Q. It follows from § 2, Lemma 1 that R is a complement of Q in G^*. Since Q is part of the abelian group P, we deduce from Dedekind's Law that P is the direct product of Q and $P \cap R$. Hence $Q = Q \cap P^p = Q^p$, an impossibility since Q is a finite abelian p-group. Thus $P^p = 1$; and this shows that $F(G^*) = F[G/\phi(G)]$ is an elementary abelian group.

Since the ϕ-group of an elementary abelian group is 1, we find that

$$1 = \phi(F[G/\phi(G)]) = \phi[F(G)/\phi(G)] \text{ or } \phi[F(G)] \leq \phi(G);$$

and this completes the proof.

COROLLARY 1. *The following properties of the normal subgroup N of G are equivalent.*

(i) *N is nilpotent.*

(ii) *$N/[N \cap \phi(G)]$ is an elementary abelian group.*

(iii) *$N/[N \cap \phi(G)]$ is an abelian group.*

(iv) *$N/[N \cap \phi(G)]$ is a nilpotent group.*

Proof. If N is nilpotent, then N is part of $F(G)$ [by definition]. Hence

$$N/[N \cap \phi(G)] \simeq N\phi(G)/\phi(G) \leq F(G)/\phi(G);$$

and now it follows from Theorem 1 that (ii) is a consequence of (i).

It is obvious that (ii) implies (iii) and that (iii) implies (iv).

Assume finally the validity of (iv). It follows from § 2, Corollary 1 and § 1, Proposition 3 that $N \cap \phi(G)$ is a weakly hypercentral normal subgroup of G; and consequently we may deduce from (iv) and § 1, Corollary 2 that N is nilpotent. Thus (i) is a consequence of (iv); and this completes the proof.

Remark 1. If we let in Corollary 1 in particular $N = G$, then we see that Wielandt's characterization of nilpotent groups is a special case of the preceding results; see, for instance, Zassenhaus [1; p. 108].

Remark 2. If the normal subgroup N of G satisfies $N \cap \phi(G) = 1$, then it follows from Corollary 1 that N is nilpotent if, and only if, N is an elementary abelian group.

Remark 3. Theorem 1 and Corollary 1 have also been obtained by Gaschütz [1] where different proofs of these and related results may be found.

COROLLARY 2. *If $F(G)$ is weakly hypercentral, then $G/F(G)$ does not contain soluble normal subgroups different from* 1.

Proof. Assume by way of contradiction that $F(G)$ is weakly hypercentral and that $G/F(G)$ contains a soluble normal subgroup different from 1. Then there exists a normal subgroup N of G such that $F(G) < N$ and such that $N/F(G)$ is abelian. Now we may deduce from § 1, Corollary 2 that N is nilpotent; and this is impossible, since it would imply $N \leq F(G) < N$.

Remark 4. The following fairly obvious consequence of Corollary 2 may be worth mentioning.

The group G is nilpotent if, and only if, G is soluble and $F(G)$ is weakly hypercentral.

Combining § 1, Proposition 1 and § 1, Corollary 4, § 2, Theorem 1 we obtain the following inequalities:

$$\phi(G)H(G) \leq H_w(G) \leq F(G).$$

If $F(G)$ happens to be weakly hypercentral, then $F(G)$ is the one and only one maximal weakly hypercentral normal subgroup; and now we deduce from Corollary 2 that, in general, $F(G)$ and $H_w(G)$ will be different. Furthermore it is not difficult to construct groups whose center and Frattini group equal 1, though $H_w(G)$ is different from 1. Thus $H_w(G)$ may be "somewhere" between the limits given above. But it follows from Theorem 1 that $F(G)/\phi(G)$ is an elementary abelian group, showing that the above limits are not "too far apart."

4. The automorphisms of the minimal normal subgroups. We begin by proving the following simple and important fact.

PROPOSITION 1. *$F(G)$ is part of the centralizer of every minimal normal subgroup of G.*

Proof. If M is a minimal normal subgroup of G, and if $M \cap F(G) = 1$, then every element in M commutes with every element in $F(G)$, since their commutators belong to their intersection which is 1. If next $M \cap F(G) \neq 1$, then it follows from the minimality of M that $M = M \cap F(G)$ or $M \leq F(G)$. But $F(G)$ is nilpotent and M is a normal subgroup of G. Hence $M \cap Z[F(G)] \neq 1$. As a characteristic subgroup of a characteristic subgroup $Z[F(G)]$ is a characteristic subgroup of G. Now it follows from the minimality of M that $M = M \cap Z[F(G)]$ or $M \leq Z[F(G)]$. Thus we have seen in either case that $F(G)$ is part of the centralizer of M.

If N is a normal subgroup of G, then the centralizer $Z(N<G)$ of N in G is likewise a normal subgroup of G and $G/Z(N<G)$ is essentially the same as the group of automorphisms of N which are induced in N by inner automorphisms of G. If S is a subgroup of G such that $G=SZ(N<G)$, then every automorphism of N which is induced by an inner automorphism of G is also induced in N by an element of S; and for this reason we shall say that *G is represented in its normal subgroup N by its subgroup S* whenever $G=SZ(N<G)$. There always exist subgroups of G which represent G in N, for instance G itself. Consequently there exist also minimal subgroups representing G in N; and these will be of particular interest to us.

LEMMA 1. *Assume that M is a minimal normal subgroup of G and that G is represented in M by its subgroup S. Then*

(a) $M\leq S$ *or* $M\cap S=1$;

(b) $S\cap Z(M<G)$ *is a normal subgroup of* MS;

(c) M *is a minimal normal subgroup of* MS.

Proof. If $M\cap S\neq 1$, then $M\cap S$ contains an element $x\neq 1$. But the same elements are conjugate to x in G and in S, since $G=SZ(M<G)$. Consequently $M\cap S$ contains a full set of elements conjugate to x in G. Since M is a minimal normal subgroup of G, M is generated by any one of its elements, not 1, together with its conjugates; and now it is clear that $M\cap S\neq 1$ implies $M\leq S$. By essentially the same argument one sees that M is a minimal normal subgroup of MS.

$S\cap Z(M<G)$ is a normal subgroup of S, since $Z(M<G)$ is a normal subgroup of G. Thus elements in S transform $S\cap Z(M<G)$ into itself. Elements in M commute with every element in $S\cap Z(M<G)$; and now it is clear that $S\cap Z(M<G)$ is a normal subgroup of MS.

LEMMA 2. *Assume that the minimal normal subgroup M of G is abelian and that S is a minimal subgroup representing G in M. Then*

(a) MS *is a minimal subgroup T with the properties:* $M\leq T$ *and* $G=TZ(M<G)$;

(b) $S\cap Z(M<G)\leq\phi(S)$;

(c) $F(MS)=M[Z(M<G)\cap S]=Z(M<G)\cap MS$.

Proof. It is clear that $M\leq MS$; and from the commutativity of M we infer $G=SZ(M<G)=[SM]Z(M<G)$. Assume next that Q is a subgroup with the following properties:

$$M\leq Q\leq MS \text{ and } G=QZ(M<G).$$

Then it follows from Dedekind's Law and the normality of M that $Q = M(Q \cap S)$. From $M \leq Z(M < G)$ we deduce now that

$$G = QZ(M < G) = (Q \cap S)MZ(M < G) = (Q \cap S)Z(M < G).$$

Hence $Q \cap S$ represents G in M; and now it follows from the minimality of S that $S = Q \cap S$ or $S \leq Q$. Hence $MS \leq Q \leq MS$ or $Q = MS$ proving the desired minimality of MS.

$J = S \cap Z(M < G)$ is a normal subgroup of S. Suppose now that H is some subgroup of S which satisfies $JH = S$. Then we have

$$HZ(M < G) = HJZ(M < G) = SZ(M < G) = G,$$

since J is part of $Z(M < G)$ and a normal subgroup of S. Hence H represents G in M and we deduce $H = S$ from the minimality of S. Thus J is a normal subgroup of S which satisfies condition (ii) of § 2, Proposition 1. It follows that $J \leq \phi(S)$.

The [by Lemma 1, (b)] normal subgroup J of MS is nilpotent as a subgroup of $\phi(S)$. The normal subgroup M of MS is nilpotent as an abelian group. Hence $MJ \leq F(MS)$. It follows from Lemma 1, (c) that M is a minimal normal subgroup of MS; and now it follows from Proposition 1 that

$$F(MS) \leq Z(M < MS) \leq Z(M < G).$$

Since M is abelian, $M \leq Z(M < G)$. Now we deduce from the three inequalities just derived and Dedekind's Law that

$$MJ \leq F(MS) \leq MS \cap Z(M < G) = M[S \cap Z(M < G)] = MJ;$$

and this shows the validity of the equation (c).

PROPOSITION 2. *If M is a minimal normal subgroup of G such that $G/Z(M < G)$ contains a normal subgroup, not 1, whose order is prime to the order of M, then*

(a) *M is abelian;*

(b) *every minimal subgroup S representing G in M satisfies $S \cap M = 1$ and $S \cap Z(M < G) = \phi(SM)$;*

(c) *the two minimal subgroups H and K representing G in M satisfy $MH = MK$ if, and only if, there exists an element x in M such that $H = x^{-1}Kx$.*

Proof. There exists by hypothesis a normal subgroup Q of G such that $Z(M < G) < Q$ and $[Q : Z(M < G)]$ is prime to the order of M.

Assume first by way of contradiction that $M \cap Z(M < G) = 1$. Then the order of M equals $[MZ(M < G) : Z(M < G)]$ and this number is prime to $[Q : Z(M < G)]$. The groups $MZ(M < G)/Z(M < G)$ and $Q/Z(M < G)$ have consequently relatively prime order so that their intersection is 1. Hence it follows from Dedekind's Law that

$$Z(M < G) = Q \cap MZ(M < G) = Z(M < G)[Q \cap M] \text{ or } Q \cap M \leq Z(M < G);$$

and consequently we find that $Q \cap M = Q \cap M \cap Z(M < G) = 1$, since $M \cap Z(M < G)$ is supposed to equal 1. But $Q \cap M = 1$ implies $Q \leq Z(M < G)$ contradicting $Z(M < G) < Q$. Our assumption $Z(M < G) \cap M = 1$ has therefore led us to a contradiction. Hence $Z(M < G) \cap M \neq 1$; and it follows from the minimality of M that $Z(M < G) \cap M = M$ or $M \leq Z(M < G)$; and this implies the commutativity of M.

Since M is an abelian minimal normal subgroup, there exists a uniquely determined prime number p such that $M^p = 1$ [§ 2, Remark 3].

Consider now some minimal subgroup S representing G in M. Then $G = SZ(M < G)$ and $J = S \cap Z(M < G)$ is a normal subgroup of MS [by Lemma 1, (b)] such that $J \leq \phi(S)$ [by Lemma 2, (b)]. We note furthermore that $N = Q \cap S$ is a normal subgroup of S which contains J. It follows from Dedekind's Law that

$$NZ(M < G) = [Q \cap S]Z(M < G) = Q \cap SZ(M < G) = Q \cap G = Q;$$

and now we deduce from the Isomorphism Law that

$$N/J = (S \cap Q)/(S \cap Z(M < G)) \simeq (S \cap Q)Z(M < G)/Z(M < G)$$
$$= Q/Z(M < G).$$

Thus $[N : J] \neq 1$ is prime to p, since the order of M is a power of p. Denote now by r the greatest divisor of the order of N which is prime to p. Since $[N : J]$ is prime to p, r is a multiple of $[N : J]$. From $J < \phi(S)$ we deduce the weak hypercentrality of the normal subgroup J of S [by § 2, Corollary 1 and § 1, Proposition 3]. Since N is a normal subgroup of S, we deduce from § 1, Proposition 2 that N is the direct product of an r-group N_r and a Pr-group N'_r. Since p is the only prime divisor of the order of N which is prime to r, N'_r is a p-group; and since $[N : J]$ is prime to p, we have $N'_r \leq J$. Clearly N'_r is the totality of p-elements in N and N_r is the totality of Pp-elements in N.

Assume now by way of contradiction that $M \cap S \neq 1$. Then it follows from Lemma 1, (a) that $M \leq S$. Since M is abelian, we have $M \leq Z(M < G)$;

and thus we have shown that $M \leq J$. Since N'_r is the totality of p-elements in N, we find that

$$M \leq N'_r \leq J \leq Z(M < G).$$

That N_r is part of the centralizer of M, follows from the fact that elements in N_r and in N'_r commute and that M is part of N'_r. Hence the direct product N of N_r and N'_r is part of the centralizer of M so that

$$N = N \cap Z(M < G) = S \cap Q \cap Z(M < G) = S \cap Z(M < G) = J.$$

But this contradicts the fact that $[N : J] \neq 1$ which we verified before. Thus we have been led to a contradiction which proves that $M \cap S = 1$.

Next we analyze the normal subgroup $MS \cap Q = M[S \cap Q] = MN$ of MS. We recall that N is the direct product of the p-group N'_r and the Pp-group N_r; and that $N'_r \leq J \leq Z(M < G)$. It follows that every element in N'_r commutes with every element in M and with every element in N_r. Hence M and N'_r are normal subgroups of MN so that MN'_r is a normal p-subgroup of MN. Since $MN = [MN'_r]N_r$, it follows now that MN'_r is the totality of p-elements in MN; and we may restate the results just obtained shortly in the form:

MN'_r is the p-component and N_r is a p-complement of the normal subgroup $MS \cap Q$ of MS where we term p-complement any subgroup of order prime to p whose index is a power of p. Note that N_r is not uniquely determined since it need not be normal in $MS \cap Q$.

We recall next that N_r is the totality of r-elements in the normal subgroup $N = Q \cap S$ of S. As a characteristic subgroup of a normal subgroup N_r is consequently a normal subgroup of S. In other words: S is part of the normalizer of N_r in MS. Consider next an element a belonging to the normalizer of N_r in MS. There exist elements m and s in M and S respectively such that $a = ms$. Since a belongs to the normalizer of N_r [by hypothesis] and since s belongs to the normalizer of N_r [as an element in S], m belongs to the normalizer of N_r. If x is an element in N_r, then the commutator $[x, m]$ belongs to N_r, since x is in the normalizer of N_r; and $[x, m]$ belongs to M, since M is a normal subgroup of G. Thus $[x, m]$ belongs to the intersection of M and N_r which is 1, since M is a p-group whereas N_r is a Pp-group. Hence $xm = mx$; and we have shown that m commutes with every element in N_r. Remembering that $N = N'_r N_r$, that $N'_r \leq Z(M < G)$, and that $Q = NZ(M < G) = N_r Z(M < G)$ it follows finally that m commutes with every element in Q. Hence m is a fixed element of every automorphism in $Q/Z(M < \mathrm{G})$. Since this group of auto-

morphisms of M is a normal subgroup, not 1, of the group $G/Z(M < G)$ of automorphisms of M, the group of fixed elements of $Q/Z(M < G)$ is a normal subgroup of G which is a proper part of M. It follows from the minimality of M, that this group of fixed elements equals 1. Hence $m = 1$ so that a belongs to S. Thus we have shown that

$$S \text{ is the normalizer of } N_r \text{ in } MS.$$

We are now ready to prove (c). Since the sufficiency of the condition, given in (c), is almost obvious, we assume immediately that H and K are minimal subgroups representing G in M and that $MH = MK = W$. We have shown before that the totality of p-elements in W is a characteristic subgroup W_p of W and that there exist p-complements H^* and K^* of W such that H is the normalizer of H^* in W and K is the normalizer of K^* in W. Because of the solubility of p-groups it is a special case of a Theorem of Witt-Zassenhaus that p-complements are conjugate whenever the p-Sylow subgroup is normal; see, for instance, Zassenhaus [1; p. 126, Satz 27]. Consequently H^* and K^* are conjugate in W. Hence there exists an element w in W such that $H^* = w^{-1}K^*w$. Since H and K are the normalizers in W of H^* and K^* respectively, it follows that $H = w^{-1}Kw$. Since $W = KM$, there exist elements k and x in K and M respectively such that $w = kx$. Consequently

$$H = w^{-1}Kw = x^{-1}k^{-1}Kkx = x^{-1}Kx;$$

and this completes the proof of (c).

We have already verified (a), (c) and the first part of (b); and we are going to deduce now the second part of (b) from these three properties without any further reference to the original hypotheses. Consider therefore a minimal subgroup S representing G in M. Then $M \cap S = 1$ and M is a minimal normal subgroup of MS [Lemma 1, (c)]; and S is a maximal subgroup of MS [§ 2, Lemma 1]. Consequently $\phi(MS) \leq S$. Next we deduce from the nilpotency of Frattini's subgroup and from Lemma 2, (c) that

$$\phi(MS) \leq S \cap F(MS) = S \cap Z(M < G) \cap MS = S \cap Z(M < G) = J;$$

and we deduce from Lemma 1, (b) and Lemma 2, (b) that J is a normal subgroup of MS which is part of $\phi(S)$.

Consider now some maximal subgroup T of MS. We distinguish two cases.

Case 1. $M \leq T$.

It follows from Dedekind's Law that $T = M(S \cap T)$ and that $S \cap T$ is a maximal subgroup of S. Hence $J \leq \phi(S) \leq S \cap T \leq T$.

Case 2. $M \not\leq T$.

Then $M \cap T = 1$ [§ 2, Lemma 1]; and it follows from the maximality of T that $MT = MS$. Next it is clear that

$$TZ(M < G) = TMZ(M < G) = SMZ(M < G) = SZ(M < G) = G;$$

T represents G in M. Suppose now that the subgroup X of T represents G in M. Then $G = [XM]Z(M < G)$ and $XM \leq TM = SM$ imply, by Lemma 2, (a), that $XM = SM = TM$ and that therefore $X = T$. In other words: T is a minimal subgroup representing G in M. But then we deduce from (c) the existence of an element v in M such that $T = v^{-1}Sv$. Consequently

$$J = v^{-1}Jv \leq v^{-1}Sv = T.$$

Thus we have shown that J is part of every maximal subgroup T of MS. Hence $J \leq \phi(MS)$; and this completes the proof of the equation

$$\phi(MS) = J = Z(M < G) \cap S.$$

Remark. The principal content of the preceding discussion for our future applications may be stated as follows:

(E) *If M is a minimal normal subgroup of G, and if $G/Z(M < G)$ contains a normal subgroup, not 1, whose order is prime to the order of M, then M is abelian and there exists a subgroup S of G such that $M \cap S = 1$ and $SZ(M < G) = G$.*

It is natural that this fundamental existence theorem (E) ceases to be true once we omit the hypothesis concerning $G/Z(M < G)$. The author is indebted to Professor Wielandt for pointing out to him the impossibility of substituting for the hypothesis concerning $G/Z(M < G)$ the weaker assumption that M be abelian.

5. Solubility. The following simple facts will be needed to obtain the connection between the results of the last section and the solubility problem.

LEMMA 1. *If the minimal normal subgroup M of G is soluble, then M is abelian and there exists a prime p such that $M^p = 1$.*

Proof. The commutator subgroup $[M, M]$ of M is a normal subgroup of G, since it is a characteristic subgroup of a normal subgroup. Furthermore $[M, M] < M$, since M is soluble. Hence $[M, M] = 1$ so that M is abelian. The existence of a prime p such that $M^p = 1$ may finally be deduced from § 2, Remark 3.

LEMMA 2. *If M is a minimal normal subgroup of G, and if $G/Z(M<G)$ contains a soluble normal subgroup different from 1, then [M is abelian and] $G/Z(M<G)$ contains a normal subgroup different from 1 whose order is prime to the order of M.*

Proof. We deduce from our hypothesis the existence of a minimal normal subgroup Q of $G/Z(M<G)$ which is soluble. It follows from Lemma 1 that Q is abelian and that there exists a prime number q such that $Q^q=1$.

Assume now by way of contradiction that M is not part of $Z(M<G)$. Then $M\cap Z(M<G)\neq M$ and this implies $M\cap Z(M<G)=1$, since M is a minimal normal subgroup of G. Consequently $M^*=MZ(M<G)/Z(M<G)$ is a minimal normal subgroup of $G^*=G/Z(M<G)$. We distinguish two cases.

Case 1. $M^*\cap Q\neq 1$.

Then it follows from the minimality of the normal subgroups M^* and Q of G^* that $M^*=Q$. Hence M^* is soluble. But $M^*=MZ(M<G)/Z(M<G)$ is isomorphic to M, since $M\cap Z(M<G)=1$. Thus M is soluble. But then we infer from the minimality of M and Lemma 1 that M is abelian. Hence M is part of $Z(M<G)$ contradicting $M\cap Z(M<G)=1$. Thus this case 1 is impossible.

Case 2. $M^*\cap Q=1$.

There exists a uniquely determined normal subgroup R of G which contains $Z(M<G)$ and satisfies $Q=R/Z(M<G)$. The hypothesis of case 2 is then equivalent to

$$Z(M<G)=MZ(M<G)\cap R=Z(M<G)[M\cap R],$$

as follows from Dedekind's Law. Consequently $M\cap R$ is part of $Z(M<G)$. Hence

$$M\cap R=M\cap R\cap Z(M<G)=M\cap Z(M<G)=1.$$

Since M and R are both normal subgroups, it follows now that every element in M commutes with every element in R. Consequently $R\leq Z(M<G)$ which contradicts $Q\neq 1$. Thus we have been led again to a contradiction. Consequently M is part of $Z(M<G)$; and this implies in particular that M is abelian. We deduce from Lemma 1 the existence of a prime p such that $M^p=1$.

Assume now by way of contradiction that $p=q$. Then Q is essentially the same as a p-group of automorphisms of the p-group M. Denote by N the group of all the fixed elements of this group Q of automorphisms of M.

Since Q is a normal subgroup of $G/Z(M < G)$, N is a normal subgroup of G. Since $Q \neq 1$, $N < M$. It follows from the minimality of M that $N = 1$. But a p-group of automorphisms of a p-group always possesses fixed elements different from 1, as may be seen from the customary arguments [like those employed when proving that the center of a p-group, not 1, is different from 1]. Thus we have have been led to a contradiction. Hence $p \neq q$ and this completes the proof.

LEMMA 3. *If $\phi(G)$ contains every normal subgroup of G except G, and if $G/\phi(G)$ is not abelian, then center and hypercenter of G are equal subgroups of $\phi(G) = F(G)$.*

Proof. G is not nilpotent, since $G/\phi(G)$ is not abelian [Wielandt's Theorem; see, for instance, § 3, Corollary 1]. Consequently $\phi(G) = F(G)$ and the center $Z(G)$ and the hypercenter $H(G)$ of G are both part of $\phi(G)$.

Suppose now that the normal subgroup N of G contains $Z(G)$ and that $N/Z(G) \leq Z[G/Z(G)]$. Then N is part of $H(G)$ so that $N \leq \phi(G)$. If n is an element in N and g an element in G, then their commutator $[n, g] = n^{-1}g^{-1}ng$ belongs to $Z(G)$. If g and h are elements in G, and if n is an element in N, then we find that

$$(gh)^{-1}n(gh) = h^{-1}g^{-1}ngh = h^{-1}n[n, g]h = h^{-1}nh[n, g]$$
$$= n[n, h][n, g] = n[n, g][n, h] = (hg)^{-1}n(hg).$$

The automorphisms which are induced in N by elements in G form therefore a commutative group; and this is equivalent to saying that $G/Z(N < G)$ is abelian. Since $G/\phi(G)$ is not abelian, it is impossible that $Z(N < G)$ is part of $\phi(G)$. Hence $G = Z(N < G)$ or $N \leq Z(G)$. Consequently $N = Z(G)$; and now the equality of $Z(G)$ and $H(G)$ is easily deduced.

THEOREM 1. *The following properties of the group G are equivalent.*

(i) *G is soluble.*

(ii) *$G/\phi(G)$ is soluble.*

(iii) *If $Q \neq 1$ is a quotient group of G, then $\phi(Q) < F(Q)$.*

(iv) *If $Q \neq 1$ is a quotient group of G, then $F(Q) \neq 1$.*

(v) *If M is a minimal normal subgroup of the quotient group Q of G, and if $Z(M < Q) < Q$, then there exists a normal subgroup, not 1, of $Q/Z(M < Q)$ whose order is prime to the order of M.*

(vi) *If M is a minimal normal subgroup of the quotient group Q of*

G, then there exists a subgroup S of Q such that $Q = SZ(M < Q)$ and $1 = S \cap M$.

(vii) *If the normal subgroup $M \neq 1$ of the quotient group Q of G is part of every proper normal subgroup of Q, then $Z(M < Q) \neq 1$.*

(viii) *Every subgroup $S \neq 1$ of G has the following two properties.*

(a) $Z[F(S)] = Z[F(S) < S]$.

(b) *If $\phi(S)$ contains every normal subgroup of S, except S, and if $S/\phi(S)$ is not abelian, then $\phi(S)$ is the hypercenter of S.*

(ix) *If $S \neq 1$ is a subgroup of G, and if $\phi(S)$ contains every normal subgroup of S, except S, then $S/\phi(S)$ is abelian.*

(x) *If $S \neq 1$ is a subgroup of G, then $\phi(S) < F(S)$.*

Proof. The equivalence of properties (i) and (ii) is a fairly immediate consequence of the following facts: $\phi(G)$ is nilpotent and therefore soluble; if N is a normal subgroup of G, then solubility of N and G/N is necessary and sufficient for solubility of G.

If G is soluble and if $Q \neq 1$ is a quotient group of G, then Q too is soluble. From $Q \neq 1$ we infer $\phi(Q) < Q$. Hence $Q/\phi(Q)$ is a soluble group different from 1. But such a group contains abelian normal subgroups different from 1. It follows from § 3, Theorem 1 that

$$1 \neq F[Q/\phi(Q)] = F(Q)/\phi(Q) \text{ or } \phi(Q) < F(Q).$$

Thus (iii) is a consequence of (i); and it is obvious that (iv) is a consequence of (iii).

Assume next the validity of (iv). The group G possesses soluble normal subgroups, for instance 1; and consequently there exists a maximal soluble normal subgroup M of G. If M were different from G, then we would infer from (iv) that $F(G/M) \neq 1$. There exists one and only one normal subgroup N of G such that $M < N$ and $N/M = F(G/M)$. From the solubility of M and $F(G/M) = N/M$ we deduce now the solubility of N. This contradicts the maximality of M. Hence $M = G$ is soluble; and we have verified the equivalence of the first four conditions.

If G is soluble, and if M is a minimal normal subgroup of the quotient group Q of G such that $Z(M < Q) < Q$, then $Q/Z(M < Q)$ is soluble as a homomorphic image of the soluble group G. From $Q/Z(M < Q) \neq 1$ and Lemma 2 we deduce now the existence of a normal subgroup, not 1, of $Q/Z(M < Q)$ whose order is prime to the order of M. Thus (v) is a consequence of (i).

Assume next the validity of (v) and consider a minimal normal subgroup M of the quotient group Q of G. If $Z(M < Q) = Q$, then $S = 1$ meets the requirements of (vi). If on the other hand $Z(M < Q) < Q$, then it follows from (v) that $Q/Z(M < Q)$ contains a normal subgroup, not 1, whose order is prime to the order of M; and we deduce from § 4, Proposition 2 [or § 4, (E)] the existence of a subgroup S of Q such that $Q = SZ(M < Q)$ and $1 = M \cap S$. Hence (vi) is a consequence of (v).

Assume next the validity of (vi) and consider a normal subgroup $M \neq 1$ of the quotient group Q of G such that every proper normal subgroup of Q contains M. Then M is a minimal normal subgroup of Q; and we deduce from (vi) the existence of a subgroup S of Q such that $Q = SZ(M < Q)$ and $1 = M \cap S$. If $Z(M < Q)$ were equal to 1, then S would equal Q and this would imply $M = 1$, an impossibility. Hence $Z(M < Q) \neq 1$ so that (vii) is a consequence of (vi).

Assume next the validity of (vii) and assume by way of contradiction that G is not soluble. Then G possesses normal subgroups V such that G/V is not soluble, for instance $V = 1$; and among these normal subgroups there exists a maximal one W. Then $Q = G/W$ is not soluble; but if $U \neq 1$ is a normal subgroup of Q, then Q/U is soluble. Clearly $Q \neq 1$ and consequently there exists a minimal normal subgroup M of Q. From $M \neq 1$ we infer the solubility of Q/M. Now we distinguish two cases.

Case 1. There exists a proper normal subgroup N of Q which does not contain M.

From $N \neq 1$ we infer the solubility of Q/N. Since the minimal normal subgroup M of Q is not part of N, we have $M \cap N = 1$. Hence M is isomorphic to the subgroup NM/N of the soluble group Q/N. Consequently the minimal normal subgroup M is soluble; and it follows from Lemma 1 that M is abelian.

Case 2. M is part of every proper normal subgroup of Q.

Then we apply (vii) and find that $Z(M < Q) \neq 1$. Hence M is part of $Z(M < Q)$; and we see again that M is abelian.

Thus we have shown that M and Q/M are both soluble; and this proves the solubility of Q, providing us with the desired contradiction. Hence G is soluble; and we have verified the equivalence of the first seven conditions.

Assume again the solubility of G and consider a subgroup $S \neq 1$ of G. Then S too is soluble. This implies in particular the solubility of $Z[F(S) < S]/Z[F(S)]$; and it follows from § 3, Proposition 2, that

$Z[F(S)] = Z[F(S) < S]$. Assume next that $\phi(S)$ contains every normal subgroup of S, except S. Since S is soluble, it follows from (iii) that $\phi(S) < F(S)$. Hence $S = F(S)$ is nilpotent and this implies commutativity of $S/\phi(S)$; see, for instance § 3, Corollary 1. Thus (viii) is a consequence of (i).

Assume next the validity of (viii) and consider a subgroup $S \neq 1$ of G such that $\phi(S)$ contains every normal subgroup of S except S. If $S/\phi(S)$ were not abelian, then $\phi(S)$ would, by (viii. b), be the hypercenter of S. But now we could infer from Lemma 3 that $\phi(S)$ is the center of S. Since $S/\phi(S)$ is not abelian, it is furthermore impossible that $S = F(S)$ [§ 3, Theorem 1]. Hence $F(S) = \phi(S)$ is the center of S so that

$$Z[F(S)] \leq F(S) = \phi(S) < S = Z[F(S) < S],$$

contradicting (viii. a). Hence $S/\phi(S)$ is abelian; and we have shown that (ix) is a consequence of (viii).

Assume next the validity of (ix). If G were not soluble, then there would exist a minimal subgroup S of G such that S is not soluble. Thus S itself is not soluble, but every proper subgroup of S is soluble. Consider now a normal subgroup N of S which is different from S and not part of $\phi(S)$. Then $N < S$ implies the solubility of N. Since N is not part of $\phi(S)$, there exists a maximal subgroup T of S which does not contain N. Clearly T is soluble and $S = NT$. But then $S/N \simeq T/(T \cap N)$ is soluble as a homomorphic image of the soluble group T. The solubility of N and S/N implies the solubility of S which is impossible. Thus we have shown that every normal subgroup of S with the exception of S itself is part of $\phi(S)$; and now it follows from (ix) that $S/\phi(S)$ is abelian. But $\phi(S)$ is nilpotent and consequently soluble proving again the solubility of S. Our hypothesis that G is not soluble has led us to a contradiction which proves the equivalence of properties (i) to (ix).

If G is soluble, then every subgroup $S \neq 1$ of G is soluble too; and $\phi(S) < F(S)$ may be derived from (iii). Thus (x) is a consequence of (i). Assume conversely the validity of (x) and consider a subgroup $S \neq 1$ of G such that every normal subgroup, not S, of S is part of $\phi(S)$. It follows from (x) that $\phi(S) < F(S)$ and that therefore $F(S) = S$. The commutativity of $S/\phi(S) = F(S)/\phi(S)$ is now a consequence of § 3, Theorem 1. Hence (ix) is a consequence of (x); and this completes the proof of the equivalence of our ten properties.

Remark. The author is indebted to Professor Wielandt for pointing out to him a class of groups which shows the impossibility of essentially weakening

condition (viii). This is the more interesting, since one may deduce from (ix) the absence of subgroups S of soluble groups G with the following properties: $\phi(S)$ contains every proper normal subgroup of S and $S/\phi(S)$ is not abelian. Hence (viii. b) is somewhat vacuously satisfied by soluble groups.

6. n-soluble groups. For a convenient enunciation of our subsequent results we need the following concepts which have been introduced elsewhere; see Baer [3, 4]. The element z in the group G is termed *n-central*, for n an integer, if

$$(zg)^n = z^n g^n \text{ and } (gz)^n = g^n z^n \text{ for every } g \text{ in } G;$$

and the totality of n-central elements in G is the *n-center* $Z(G;n)$ of G. The n-center is a characteristic subgroup and $Z(G;n) = Z(G;1-n)$. Similarly we term the group G *n-abelian*, if

$$(xy)^n = x^n y^n \text{ for every } x \text{ and } y \text{ in } G.$$

It has been shown in Baer [4] that the study of n-abelian groups may effectively be reduced to the study of n-abelian n-groups.

PROPOSITION 1. *If G is an n-group and $\phi(G) = 1$, then the following properties of G are equivalent.*

(i) *G is n-abelian.*

(ii) *Every minimal normal subgroup of G belongs to the n-center of G.*

(iii) $G^n = 1$.

Proof. It is trivial that (i) implies (ii) and that (iii) implies (i). Thus we need show only that (iii) is a consequence of (ii). Assume consequently the validity of (ii); and assume by way of contradiction that $G^n \neq 1$. Then there exists a minimal normal subgroup M of G which is part of G^n; and we deduce $M \leq Z(G;n)$ from (ii). Since G is an n-group, so is M; and this implies that every element in M is the $(1-n)$-th power of an element in M. But the $(1-n)$-th powers of elements in the n-center commute with every n-th power; see Baer [3; (2. 4, b)]. Consequently G^n is part of the centralizer of M in G so that

$$M \leq G^n \leq Z(M < G).$$

Hence M is abelian; and we deduce from the minimality of M the existence of a prime p such that $M^p = 1$ [§ 5, Lemma 1]. Since M is an n-group, p is a divisor of n. From $\phi(G) = 1$ we deduce the existence of a maximal

subgroup S of G which does not contain the abelian minimal normal subgroup M of G. It follows from § 2, Lemma 1 that S is a complement of M in G. Consider now an element g in G. Then there exist uniquely determined elements m and s in M and S respectively such that $g = ms$. Since m, as an element in $M \leq Z(G; n)$, is n-central, we find that $g^n = (ms)^n = m^n s^n = s^n$. [Remember that $M^p = 1$ and that p is a divisor of n.] Hence $G^n \leq S$ so that $1 < M = M \cap G^n \leq M \cap S = 1$, a contradiction which proves $G^n = 1$. This completes the proof.

COROLLARY 1. *The group G without proper characteristic subgroups is n-abelian if, and only if, G is abelian or $G^n = 1$ or $G^{1-n} = 1$.*

Proof. The sufficiency of our condition is almost obvious. Assume therefore that G is n-abelian. Then it is known that G is the direct product of an n-abelian n-group G_n, a $(1-n)$-abelian $(1-n)$-group G_{1-n} and of an abelian $Pn(1-n)$-group A; see Baer [4]. It is clear that G_n, G_{1-n} and A are characteristic subgroups and that therefore only one of them may be different from 1. It follows furthermore from $\phi(G) < G$ and the absence of proper characteristic subgroups that $\phi(G) = 1$. Now we deduce $G^n = 1$ from Proposition 1, in case $G = G_n$; and we deduce $G^{1-n} = 1$ from Proposition 1, in case $G = G_{1-n}$. This completes the proof.

A group G has been termed *n-soluble*, if every quotient group $Q \neq 1$ of G possesses an n-abelian minimal normal subgroup; see Baer [4]. This is clearly equivalent to the requirement that every principal factor of G is n-abelian. [For the definition of principal factors see Zassenhaus [1; p. 53].] Thus we obtain the following consequence of Corollary 1.

PROPOSITION 2. *The group G is n-soluble if, and only if, every principal factor C of G satisfies one of the following three conditions:*

$$C \text{ is abelian or } C^n = 1 \text{ or } C^{1-n} = 1.$$

7. n-nilpotent n-groups. A group G has been termed *n-nilpotent*, if the n-center of each of its quotient groups, not 1, is different from 1; see Baer [4]. It is not difficult to see that n-nilpotent groups have the following stronger property:

(N-n) *If M is a minimal normal subgroup of the quotient group Q of G, then M is part of the n-center of Q.*

It has been shown in Baer [4] that the study of n-nilpotent groups may

effectively be reduced to the study of n-nilpotent n-groups and thus these will be the object of our present investigation.

PROPOSITION 1. *If the n-group G is n-nilpotent, then $G^n \leq \phi(G)$.*

Proof. Let $G^* = G/\phi(G)$. Then G^* is an n-nilpotent n-group such that $\phi(G^*) = 1$. Hence it follows from (N-n) and § 6, Proposition 1 that $G^{*n} = 1$; and this is equivalent to $G^n \leq \phi(G)$.

Remark 1. It is not difficult to prove that n-nilpotency of a random group implies n-commutativity of $G/\phi(G)$.

PROPOSITION 2. *The soluble n-group G is n-nilpotent if, and only if,*

$$S^n \leq \phi(S) \text{ for every subgroup } S \text{ of } G.$$

Proof. Since every subgroup of an n-nilpotent n-group is itself an n-nilpotent n-group, we may derive the necessity of our condition from Proposition 1.

Assume conversely the validity of our condition. Consider a quotient group $Q = G/N \neq 1$ of G and a minimal normal subgroup $M/N = M^*$ of Q. From the solubility of G, and hence of Q, we deduce, by § 5, Theorem 1, (vi) the existence of a subgroup S^* of Q such that $1 = M^* \cap S^*$ and $Q = S^* Z(M^* < Q)$. There exist uniquely determined subgroups S and V of G which both contain N and satisfy $S^* = S/N$ and $Z(M^* < Q) = V/N$. It is clear that V is a normal subgroup of G and that $N = M \cap S$ and $G = VS$.

Next we let $T = MS$ and $T^* = T/N = M^* S^*$. Since the elements in S^* induce in M^* all the automorphisms that are induced in M^* by elements in G^*, M^* is likewise a minimal normal subgroup of T^*. Since S^* is a complement of M^* in T^*, it follows from § 2, Lemma 1 that S^* is a maximal subgroup of T^*. Hence S is a maximal subgroup of T; and it follows now from our condition that $T^n \leq \phi(T) \leq S$, and that consequently $T^{*n} \leqq S^*$. Consider now an element t^* in T^* and an element m^* in M^*. Then $(t^* m^*)^n (t^{*n} m^{*n})^{-1}$ and $(m^* t^*)^n (m^{*n} t^{*n})^{-1}$ belong to T^{*n} and therefore to S^*; and they also belong to M^*, since M^* is a normal subgroup of T^*. These elements belong therefore to the intersection 1 of M^* and S^*; and this shows that

$$(t^* m^*)^n = t^{*n} m^{*n} \text{ and } (m^* t^*)^n = m^{*n} t^{*n} \text{ for } m^* \text{ in } M^* \text{ and } t^* \text{ in } T^*.$$

Consider now an element m^* in M^* and an element q in Q. Then there exists an element s^* in S^* such that $q \equiv s^*$ modulo $Z(M^* < Q)$. The

elements q and s^* induce therefore the same automorphism in M^* and this implies in particular

$$q^{-i}m^*q^i = s^{*-i}m^*s^{*i} \text{ for every integer } i.$$

Application of this and the preceding formula gives us now:

$$\begin{aligned}(m^*q)^n &= m^*(qm^*q^{-1})(q^2m^*q^{-2})\cdots(q^{n-1}m^*q^{1-n})q^n \\ &= m^*(s^*m^*s^{*-1})(s^{*2}m^*s^{*-2})\cdots(s^{*n-1}m^*s^{*1-n})q^n \\ &= (m^*s^*)^n s^{*-n}q^n = m^{*n}q^n;\end{aligned}$$

and $(qm^*)^n = q^nm^{*n}$ is shown likewise. Thus we have finally verified that M^* is part of the n-center of Q. We have therefore proven the validity of (N-n) and consequently the n-nilpotency of G.

Remark 2. The author has not been able to decide whether the solubility hypothesis may be omitted from Proposition 2 nor could he decide whether the weaker hypothesis $G^n \leqq \phi(G)$ is sufficient.

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